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# RATE OF CONVERGENCE, RELATION BETWEEN THE ARCSINE LAW AND P($\tau \leq n$) IN THE OPTIMAL STOPPING PROBLEM FOR $\frac{S_n}{n}$

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**ABSTRACT**: *We establish the relation between the arcsine law and P($\tau \leq n$), and we prove that* $Sup_{t \leq N} E \frac{S_t}{t} \to Sup_{t \leq \infty} E \frac{S_t}{t}$ *as* $N \to \infty$ *and the rate of convergence belongs between* $\frac{1}{2(N+1)} E \ |X_1| \ \forall N$ *and* $(1+\varepsilon)\frac{2}{N^{\frac{3}{4}}}$ *for arbitrary* $\varepsilon > 0$, $N > N_\varepsilon$ .

**KEY WORDS**: Optimal stopping time, backward induction, the arcsine law, the Fatou – Lebesgue`s theorem. Symmetric Bernoulli`s distribution. Generating function, AMS (2000) Subject classification. Primary 60G 40. Secondary 62L 15.

## INTRODUCTION

In (Chow, Y.S., Robbins, H., 1965) the existence of an optimal stopping time for $\frac{S_t}{t}$ was proved. The case P(t=∞) was considered by Klass, M, J (1973). In (Lai, T., L., and Yao, Y.C., 2005) it was proved that: $0 \leq V(x, n) - V_N (x, n) \leq \frac{b_N}{N} \leq \frac{0,83992}{\sqrt{N}}$ where $V(x,n) = Sup_{t \in T} E(\frac{x+S_t}{n+t})$, T is the set of all stopping times, $V_N(x,n)$ is the value function of the finite horizon problem, moreover, authors give the formula: $0 \leq V(x,n) - V_N (x, n) \leq \frac{b_N}{N}$ . $P(x+S_k < b_{n+k}$, k= 1,…, (N-n), but the value of P was not given and there was not result for the case of x = 0, n = 0.

In this work, by the structure of the optimal stopping time, the relation between the arcsine law and P($\tau \leq n$) , we have the following results:

a) $Sup_{t \leq \infty} E \frac{S_t}{t} = \lim_{N \to \infty} Sup_{t \leq N} E \frac{S_t}{t}$,

b) $Sup_{t \leq \infty} E \frac{S_t}{t} - Sup_{t \leq N} E \frac{S_t}{t} \leq (1+\varepsilon) \frac{2}{N^{3/4}}$, for arbitrary $\varepsilon > 0$ and $N > N_\varepsilon$.

c) $Sup_{t \leq \infty} E \frac{S_t}{t} - Sup_{t \leq N} E \frac{S_t}{t} \geq \frac{E|X_1|}{2(N+1)}$. $\forall$ N.



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2) ( $V^{N+1}$- $V^N$ ) and V = $V^\infty$.

Let $X_i$ i = 1, 2,… be independent identically distributed (i. i. d) random variables having $EX_i = 0$, $VarX_i = 1$. Let $S_n = X_1 + X_2 + \dots + X_n$. We suppose that $F_n$ is the σ-algebra generated by $X_1,\dots,X_n$ , t is a stopping time related to ($F_n$, n= 1, 2, …), T is the set of all stopping times, $T^N$ is the set of stopping times such that t ≤ N. A stopping time $\tau \in T^N$ is said to be optimal in $T^N$ if:

$$E\,\frac{S_\tau}{\tau} = Sup_{t\in T^N}\, E\,\frac{S_t}{t}\,. \tag{2.1}$$

In all this work we put:

$$V = Sup_{t\le\infty}\, E\,\frac{S_t}{t}\ ,\ V^N = Sup_{t\le N}\, E\,\frac{S_t}{t}\ , V^\infty = Lim_{N\to\infty} V^N. \tag{2.1a}$$

THEOREM (2.1)

Suppose that $\tau$ is optimal in $T^{N+1}$, μ is optimal in $T^N$ then:

$$E\,\frac{S_\tau}{\tau} - E\,\frac{S_\mu}{\mu} \le \int_{\Delta_{N+1}} \frac{-S_N}{(N+1)N}\, dP,\ \Delta_{N+1} = \{\tau = N+1\}. \tag{2.2}$$

PROOF

We construct the stopping $t \in T^N$ such that :

$$\{t=n\} = \{\tau = n\},\ n = 1, 2, \dots (N-1),\ \{t=N\} = R^N \backslash \textstyle\sum_1^{N-1}\{\tau = n\},$$

hence we have:

$$E\,\frac{S_\tau}{\tau} - E\,\frac{S_t}{t} = \int_{\{\tau=N+1\}} \frac{X_{N+1}}{(N+1)}\ dP - \int_{\{\tau=N+1\}} \frac{S_N}{(N+1)N}\ dP\ , \tag{2.3}$$

$\{\tau = N+1\}$ is a cylinder in $R^{N+1}$:

$\{\tau = N+1\} = \{R^N \backslash \sum_1^N\{\tau = n\}\} \times \{-\infty \le X_{N+1} \le +\infty\}$ then $\int_{\{\tau=N+1\}} \frac{X_{N+1}}{(N+1)}\, dP =$

$\int \dots \int_{\{R^N \backslash \sum_1^N (\tau=n)\}} dP_{X_1} \dots dP_{X_N} \times \int_{-\infty}^{+\infty} \frac{X_{N+1}}{(N+1)} dP_{X_{N+1}} = 0$ .(Loe`ve, M., 1960. Chapters 2, 5, 7) (Neveu, J., 1964 Chapter 3)

By (2.3) and by μ is optimal in $T^N$ the proof is complete.

THEOREM (2.2)



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$$V = V^{\infty}. \qquad (2.4)$$

PROOF

Suppose (2.4) is not true then there exists a stopping time $\tau$ such that:

$$E\,\frac{S_\tau}{\tau} = \sum_1^{\infty} \int_{\{\tau=n\}} \frac{S_n}{n}\, dP \;>\; V^{\infty} + \;\varepsilon\;. \qquad \varepsilon \;>\; 0 \qquad (2.5)$$

We construct a stopping time $\rho \leq$ N such that:

$\{\rho = n\} = \{\tau = n\}$, n = 1, 2, … , (N-1), $\{\rho = N\} = R^N \setminus U_1^{N-1}\{\tau = n\}$ and we have:

$$E\,\frac{S_\tau}{\tau} \leq E\,\frac{S_\rho}{\rho} + \left|\int_{\{\rho=N\}} \frac{S_N}{N}\, dP\right| + \sum_{N}^{\infty} .\int_{\{\tau=n\}} .\frac{S_n}{n}\, dP.$$

By the strong law of large numbers and by the Fatou- Lebesgue`s theorem there exists a number $N_\varepsilon$ such that for N ≥ $N_\varepsilon$ we obtain:

$$E\,\frac{S_\tau}{\tau} \leq V^{\infty} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\;. \qquad (2.6)$$

(2.6) contradicts (2.5) and the proof is complete.

3). Relation between the arcsine law and P $(\tau \leq n)$, the rate of convergence.

3.1. Let $X_i$ i = 1, 2, … be (i.i.d) random variables having continuous, symmetric distribution with

$EX_i = 0$, Var $X_i = 1$. Let $S_n = X_1 + X_2 + \ldots + X_n$.

Let K be the random variable defined by:

$\{K = k\} = \{K_N = k\} = \{ S_k > S_0, \ldots, S_k > S_{k-1}, S_k \geq S_{k+1}, \ldots, S_k \geq S_N\}$, k = 0,1,...,N.

$(S_0=0)$ (3.1) Let the arcsine law: $lim_{n\to\infty}$ P$(K < n\alpha) = \frac{2}{\pi}\arcsin\sqrt{\alpha}$ , $0 < \alpha < 1$, $\alpha$ = const. (Feller, W., Vol. II, Chapter XII, 1966).

THEOREM (3.1)

Let τ be the optimal stopping time in$T^N$,

a) If there exists $(\tau = m)$ then: P $(\tau \leq m) \geq$ P $(1 \leq K \leq m)$. $1 < m < N$. (3.2)



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b) $P(\tau > m) \leq P(K > m) + P(S_1 \leq 0; \ldots; S_N \leq 0)$. $\quad 0 \leq m \leq N-1$.
(3.3)

PROOF

By definition and by contradiction it is easy to prove that: $\{S_1; \ldots; S_N\} = R^N \setminus \{1 \leq K \leq N\}$ if and only if : $S_1 \leq 0; \ldots; S_N \leq 0$, then :

$$P(1 \leq K \leq N) + P(S_1 \leq 0; \ldots; S_N \leq 0) = 1$$
(3.4)

By backward induction: (Chow, Y., S., Robbins, H., Siegmund, D., 1971):

$$(\tau = m) = \{ \frac{S_1}{1} < E(\gamma_2^N/F_1); \ldots; \frac{S_{m-1}}{m-1} < E(\gamma_m^N/F_{m-1}); \frac{S_m}{m} \geq E(\gamma_{m+1}^N/F_m) \}. \tag{3.5}$$

Where $\gamma_N^N = \frac{S_N}{N}$, $\gamma_n^N = \max\{ \frac{S_n}{n}; E(\gamma_{n+1}^N/F_n\}$, $n = 1,2,\ldots,(N-1)$.

$$(\tau = N) = R^N \setminus \cup_1^{N-1} (\tau = n).$$
(3.5a)

We define the random variable $K^*$:

$$\{K^* = k\} = \{S_k > 0, \frac{S_k}{k} \geq \frac{S_{k+1}}{k+1}, \ldots, \frac{S_k}{k} \geq \frac{S_N}{N}\}, \quad k=1,2,\ldots,N. \tag{3.6}$$

then $\{K = k\} \subset \{K^* = k\}$.

On $\{K^* = k\}$ we have: $\frac{S_k}{k} \geq \gamma_N^N = \frac{S_N}{N}$ (by definition) and $\frac{S_k}{k} \geq \gamma_{N-1}^N = \max(\frac{S_{N-1}}{N-1}; E(\gamma_N^N/F_{N-1}))$, (because of $\frac{S_k}{k} \geq \frac{S_{N-1}}{N-1}$ and $\frac{S_k}{k} \geq \gamma_N^N$).

Similarly, it can be shown that $\frac{S_k}{k} \geq \gamma_{N-2}^N, \ldots, \frac{S_k}{k} \geq \gamma_{k+1}^N$ and $\frac{S_k}{k} \geq E(\gamma_{k+1}^N/F_k)$, hence:

$$(K^* = k) \subset \{\frac{S_k}{k} \geq E(\gamma_{k+1}^N/F_k)\}. \tag{3.6a}$$

By (3.5):

$$(\tau > k) \subset \{\frac{S_k}{k} < E(\gamma_{k+1}^N/F_k)\}.$$
(3.7)

By (3.6a), (3.7): $(K^* = k) \cap (\tau > k) = \emptyset$, $k = 1, 2, \ldots (N-1)$, so:


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$\{K=k\} \cap (\tau > k) = \emptyset$, k = 1, 2,…, (N-1),

(3.8)

hence $(K = k) \subset (\tau \leq k)$ and $\cup_1^m$ (K= k) $\subset \cup_1^m$ $(\tau \leq k)$ and (3.2) holds. Finally, by (3.2) and (3.4) the proof is complete.

THEOREM (3.2)

If $\tau$ is optimal in $T^{N+1}$ then for an arbitrary $\varepsilon > 0$ there exists a $N_\varepsilon$ such that:

a) $P\{\tau = N+1\} \leq 2.\frac{(2N+1)!!}{(2N+2)!!} \leq (1+\varepsilon).\frac{2}{\sqrt{\pi N}}.. \ \forall\varepsilon > 0, N \geq N_\varepsilon$ .

(3.9)

b) $V^{N+1} - V^N \leq (1+\varepsilon).\frac{1,1}{N^{\frac{7}{4}}}. \quad \varepsilon > 0 \ \ N > N_\varepsilon.$

(3.10)

c) $V - V^N \leq (1+\varepsilon).\frac{2}{N^{\frac{3}{4}}}. \quad \varepsilon > 0 \ \ N \geq N_\varepsilon.$

(3.11)

PROOF

a) By (3.3) $P(\tau = N+1) \leq P(K = N+1) + P(S_1 \leq 0; \ldots; S_{N+1} \leq 0)$ .

By definition: $P(K = N+1) = P(S_{N+1} > 0; S_{N+1} > S_1; \ \ldots \ ; S_{N+1} > S_N) = p_{N+1}$.

(3.12)

The generating function G (s) of $\{p_n\}$ satisfies:

Log G(s) = $\sum_1^\infty \frac{s^n}{n} P(S_n > 0)$ , (Feller, W., Vol. II, Chap XII. 1966), for continuous and symmetric distribution Log G(s) = $\frac{1}{2}\sum_1^\infty \frac{s^n}{n}$ then $G(s) = \frac{1}{\sqrt{1-s}}$ ; -1< s<1. and:

$$p_{N+1} = \frac{(2N+1)!!}{(2N+2)!!}$$

(3.13)

$P(S_1 \leq 0; S_2 \leq 0; \ldots \ ; S_{N+1} \leq 0) = q_{N+1}$

The generating function Q(s) of $\{q_n\}$ satisfies:



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Q(s) = $\sum_1^\infty \frac{s^n}{n}$ P($S_n \leq 0$ ) (Feller, W., Vol II, Chapter XII , 1966). For continuous and symmetric distribution:

$$Q(s) = \frac{1}{\sqrt{1-s}} \quad -1 < s < 1 \quad \text{and} \quad q_{N+1} = p_{N+1}. \tag{3.14}$$

By (3.13),(3.14) we have: P( $\tau$ = N+1) $\leq$ 2. $\frac{(2N+1)!!}{(2N+2)!!}$ . By the Wallis' formula we have

P($\tau$ = N+1) $\leq$ $\frac{2}{\sqrt{\pi(N+1)}}$ as N $\to \infty$ and (3,9) holds.

b) If $\tau$ *is* optimal in $T^{N+1}$ then by theorem (2.1):

$V^{N+1}$ - $V^N \leq \frac{1}{(N+1)\sqrt{N}} \int_{\Delta_{N+1}} \frac{-S_N}{\sqrt{N}}$ dP $\quad \Delta_{N+1} = \{\tau = N+1\}$,

applying the Schwarz 's inequality to the integral and by (3.9) we have (3,10).

c) Utilizing (3.10) we have $\quad V - V^N = \sum_N^\infty (V^{n+1} - V^n)$. $\quad N > N_\varepsilon$

$$V - V^N \leq 1{,}1.(\,1+\varepsilon\,).(\int_N^\infty x^{\frac{-7}{4}}\,dx + \frac{1}{N^{\frac{7}{4}}}\,) \text{ and (3.11) holds}$$

3.2 – Now we consider the case of symmetric Bernoulli' s distribution: P($X_1$= -1) = P($X_1$= 1) = $\frac{1}{2}$ .We apply the results of Sparre Andersen, E. and Spitzer., F, (Frank., Spitzer, Principles of random walk Chapter IV, 1966).

Let $N_n$ be the random variable defined by: $N_0 = 0$, $N_n = \sum_1^n \theta(S_i)$ where $\theta$(x) = 1 if x $\in$ R, x > 0, and $\theta(x) = 0$ if x$\leq$ 0 , $S_i = X_1 + \ldots + X_i$, i= 1,2,…,n. For symmetric Bernoulli's distribution Sparre., Andersen, E. proved the following results:

1) $P(N_n = k) = P(N_k = k).P(N_{n-k} = 0) \qquad 0 \leq k \leq n$

(3.15)

2) $\lim_{n\to\infty} \sqrt{n\pi}\, P(N_n = 0) = \frac{1}{\sqrt{c}}$

(3.16)

3) $\lim_{n\to\infty} \sqrt{n\pi}\, P(N_n = n) = \sqrt{c}$

(3.16a)



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Where c= $e^{-\sum_1^\infty \frac{P(S_{k=0})}{k}}$

(3.17)

THEOREM (3.1a)

Let $\tau$ be the optimal stopping time in $T^{n+1}$,

a) If there exists ($\tau$ = m), 1< m <n+1 then:

$$P(\tau \leq m) \geq P(1 \leq N_{n+1} \leq m)$$

(3.2a)

b) $P(\tau = n+1) \leq P(N_{n+1} = n+1) + P(N_{n+1} = 0)$

(3.3a)

PROOF

By (3.15), (3.1) and by duality (Feller.,W., Vol II, chapter XII, 1966 ) we have: $P(K_{n+1} = k) = P(N_{n+1} = k)$; $1 \leq k \leq n+1$. In the case of symmetric Bernoulli's distribution $\tau$ can be constructed by backward induction, then the proof of (3.2a) is like that for the proof of (3.2). By definition: $P(N_{n+1} = 0) = P(S_1 \leq 0,\dots,S_{n+1} \leq 0)$ hence $P(0 \leq N_{n+1} \leq n+1) = 1$ and (3.3a) holds.

EXAMPLE

If (n+1) =7 then: $P(\tau = 7) = \frac{30}{2^7}$, $P(N_7 = 7) = P(K_7 = 7) = \frac{20}{2^7}$.

$P(N_7 = 0) = 1 - P(1 \leq K_7 \leq 7) = \frac{35}{2^7}$. $P(\tau \leq 5) = \frac{88}{2^7} > P(K_7 \leq 5) = P(1 \leq N_7 \leq 5) = \frac{63}{2^7}$ . etc.

By (3.15), (3.16), (3.16a), similarly to the proof of theorem (3.2) we have:

THEOREM (3.2a)

If $\tau$ is optimal in $T^{n+1}$ and n→ ∞ then for an arbitrary $\varepsilon > 0$ there exists a $n_\varepsilon$ suck that:



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a) $P(\tau = n+1) \leq (1+\varepsilon)\frac{\delta^2}{\sqrt{\pi n}}$ $\qquad n > n_\varepsilon$

(3.9a)

b) $V^{n+1} - V^n \leq (1+\varepsilon)\frac{\delta}{\pi^{\frac{1}{4}} n^{\frac{7}{4}}}$ $\qquad n > n_\varepsilon$

(3.10a)

c) $V - V^n \leq (1+\varepsilon)\ \frac{2\delta}{n^{\frac{3}{4}}}$ $\qquad n > n_\varepsilon$

(3.11a)

Where $\delta^2 = (\frac{1}{\sqrt{c}} + \sqrt{c})$, c is given by (3.17).

4) The Lower rate of convergence.

Let $X_i$ , i = 1, 2, … be independent and identically distributed random variables having mean 0 and variance 1. Let $S_n = X_1 + \ldots + X_n$, n = 1, 2, …

LEMMA (4.1)

If $\tau$ is optimal in $T^N$ then;

$$\{ S_1 < 0, \ldots, S_N < 0\} \subset \{ \tau = N\}$$

(4,1)

PROOF

By the definition of $\tau$ and by $E(\gamma^N_{n+1}/ F_n) \geq E(\frac{S_{n+1}}{n+1}/F_n)$ , n = 1, 2, …., (N-1)

then: $\{\tau = n\} \subset \{\frac{S_n}{n} \geq E(\gamma^N_{n+1}/ F_n)\} \subset \{\frac{S_n}{n} \geq E(\frac{S_{n+1}}{n+1}/ F_n)\} \subset \{S_n \geq 0\}$, n = 1, …, (N-1).

Because of $\cup_1^{N-1}\{\tau = n\} \subset \cup_1^{N-1}\{S_n \geq 0\}$ then:

$\{S_1 < 0, \ldots, S_N < 0\} \subset \{S_1 < 0, \ldots, S_{N-1} < 0\} \subset \{\tau = N\}$ and the proof is complete.

LEMMA (4.2)

If $D_N = \{ S_1 < 0, \ldots, S_N < 0\}$ then:

a) $\int_{D_N} \frac{-S_N}{N}\, dP \geq \frac{1}{2N} E|X_1|, \quad N \geq 1$

(4,2)



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b) $\int_{D_N} \frac{-S_N}{N}\, dP = \frac{1}{2N}$, $N \geq 1$ for symmetric Bernoulli distribution.

(4.3)

PROOF

$a)$ $\int_{D_N} \frac{-S_N}{N} dP + \int_{CD_N} \frac{-S_N}{N} dP = \int_{D_{N-1}} \frac{-S_N}{N}\, dP$ where $CD_N = ( D_{N-1} \backslash D_N )$.

By $S_N \geq 0$ on $CD_N$; we have:

$\int_{D_N} \frac{-S_N}{N} dP \geq \int_{D_{N-1}} \frac{-S_N}{N}\, dP = \int_{D_{N-1}} E(\frac{-S_N}{N}/F_{N-1})\, dP = \frac{N-1}{N} \int_{D_{N-1}} \frac{-S_{N-1}}{N-1} dP$, then:

$\int_{D_N} \frac{-S_N}{N} dP \geq \frac{N-1}{N} \times \frac{N-2}{N-1} \times \ldots \times \frac{1}{2} E\left|\frac{X_1}{2}\right|$ . and (4.2) holds

b) For symmetric Bernoulli distribution: ($P\{X_1=1\} = P\{X_1= -1\} = \frac{1}{2}$), on $CD_N$:

$\{S_{N-1} < 0, S_N \geq 0\}$ then $S_N = 0$ and $\int_{CD_N} \frac{-S_N}{N} dP = 0$, hence (4.3) holds.

THEOREM (4.1)

$$V^{N+1} - V^N \geq \frac{1}{2(N+1)N} E|X_1| \qquad \forall\ N \geq 1 \tag{4.4}$$

PROOF

Suppose that $\tau$ is optimal in $T^{N+1}$ and $t$ is optimal in $T^N$, we construct the stopping time $\mu \in T^{N+1}$ such that:

$\{\mu = n\} = \{t = n\}$ if $n = 1, 2, \ldots, (N-1)$,

$\{\mu = N\} = \{t = N\} \setminus D_N$ where $D_N = \{ S_1 < 0, \ldots, S_N < 0 \}$.

$\{ \mu = N+1 \} = R^{N+1} \setminus \sum_1^N \{\mu = n\} = D_N \times R_{N+1} \in F_N$,

where $F_N$ is a $\sigma$ - algebra of $\Omega = R^{N+1}$.

$E\frac{S_\mu}{\mu} - E\frac{S_t}{t} = \frac{-1}{N+1} \int_{(\mu=N+1)} \frac{S_N}{N} dP + \int_{(\mu=N+1)} \frac{X_{N+1}}{N+1} dP.$

$(\mu = N+1) \in F_N$ then: $\int_{(\mu=N+1)} \frac{X_{N+1}}{N+1}\, dP = \int_{(\mu=N+1)} E(\frac{X_{N+1}}{N+1}/F_N)\, dP = 0.$

$\frac{-1}{N+1} \int_{(\mu=N+1)} \frac{S_N}{N} dP = \frac{-1}{N+1} \int_{D_N} \frac{S_N}{N} dP_{X_1} \ldots dP_{X_N} \times \int_{-\infty}^{+\infty} dP_{X_{N+1}}$ .

By lemma (4.2) and *by* $\tau$ is optimal in $T^{N+1}$, the proof is complete.



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THEOREM (4.2)

$$V \;\; - V^N \;\geq\; \frac{1}{2(N+1)}\; E|X_1| \quad \forall\; N.$$

(4.5)

PROOF
By theorem (4,1) we have:

$$V \;- V^N \geq \sum_N^{\infty} \frac{1}{2(n+1)n}\; E|X_1|. \;\geq\; E|X_1| \int_N^{\infty} \frac{1}{2(x+1)x}$$ dx and (4.5) holds.

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