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Graph of Co-Maximal Subgroups in The Integer Modulo N Group

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ABSTRACT: *This research delves into the co-maximal subgroup graph of the integer modulo n group, N_n. Investigating the structural properties of this graph provides insights into the relationships among subgroups of N_n. We explore the connectivity, patterns, and specific cases, offering a comprehensive analysis of this algebraic structure. Through a combination of group theory and graph theory, we aim to contribute to the broader understanding of subgroup interactions in cyclic groups.*

KEYWORDS: Co-maximal subgroups, integer modulo n, subgroup graph, cyclic groups, graph theory, group theory

1. INTRODUCTION

The study of subgroups within the integer modulo n group, N_n, has long been of interest in algebraic structures. This research focuses on the co-maximal subgroup graph, where vertices represent subgroups of N_n, and edges connect co-maximal subgroups. By combining group theory principles with graph-theoretic techniques, we aim to unveil the underlying patterns and characteristics of this graph. I refer the read to [1]'s book, A foundational text in finite group theory. The book provides a comprehensive introduction to group concepts relevant for understanding the structure of finite groups, including the integer modulo *n* group.

Understanding finite fields is essential for grasping the properties of N_n, see the work of [2]. [3]'s work explores applications of finite groups, while not specifically addressing N_n, it provides insights into the broader utility of group theory. Also, see [4]'s seminal work that covers a wide range of algebraic topics. The sections on group theory are particularly valuable for understanding subgroup relationships and algebraic structures, and it is crucial for grounding the study of co-maximal subgroups within a broader algebraic context.

2. PRELIMINARY

Definition 2.1. The algebraic definition of the Integer Modulo *n* Group, denoted as N_n, is as follows:
Set. The elements of N_n are equivalence classes of integers under the relation of congruence modulo *n*. Specifically, for a fixed positive integer *n*, N_n is the set of residue classes [0], [1], [2], …, [*n*−1], where each residue class represents a set of integers that have the same remainder when divided by *n*.
Group Operation (Addition)**.** The group operation in N_n is defined as modular addition. For any two elements $[a]$ and $[b]$ in N_n, the sum is given by: $[a]+[b]\equiv[a+b](\text{mod}n)$ Here, a and b are representatives of the respective residue classes.
Identity Element. The identity element in N_n is the residue class [0] since $[a]+[0]\equiv[a](\text{mod}n)$ for any $[a]$ in N_n.
Inverse Element. For each element $[a]$ in N_n, its inverse is the residue class $[-a]$ because $[a]+[-a]\equiv[0](\text{mod}n)$.
Associativity. The operation of modular addition is associative, ensuring that for any $[a]$, $[b]$, and $[c]$ in N_n, the equation $([a]+[b])+[c]=[a]+([b]+[c])$ holds.
In summary, N_n is a group under the operation of modular addition, where each element is represented by a residue class modulo *n*. This group is finite and cyclic, and its algebraic structure is essential in various mathematical applications and cryptographic systems.

Definition 2.2. Co-maximal subgroups refer to a set of subgroups within a larger group that share the property of being maximal among subgroups that do not contain the intersection of all subgroups in the set. In other words, co-maximal subgroups are maximal with respect to not containing the intersection of all subgroups in the set simultaneously.

Let's break down this definition:

1. Subgroups:
   A subgroup *H* of a group *G* is a subset of *G* that is itself a group with respect to the same group operation as *G*. Mathematically, $H \leq G$.
2. Maximal Subgroups:
   A subgroup *M* is maximal in *G* if, for any subgroup *N* such that $M \leq N \leq G$, either $N = M$ or $N = G$. Mathematically, if $M \leq N \leq G$, then either $N = M$ or $N = G$.
3. Intersection of Subgroups:
   The intersection of a set of subgroups $\{H_1, H_2, \ldots, H_k\}$ is denoted as $H_1 \cap H_2 \cap \ldots \cap H_k$. It represents the set of elements common to all subgroups in the set.
4. Co-maximal Subgroups:

- Let $\{M_1, M_2, \ldots, M_n\}$ be a set of subgroups in the group G.
- Each M_i is maximal in G (as per the definition of maximal subgroup).
- For any pair of distinct indices i and j, the intersection $M_i \cap M_j$ is not contained in any other subgroup in the set, i.e., $M_i \cap M_j \nsubseteq M_k$ for all $k \neq i, j$
- Mathematically, for all distinct indices i and j:

$$\forall\ k \neq i, j, M_i \cap M_j \nsubseteq M_k$$

This condition ensures that the intersection of any two subgroups in the set is not contained in any other subgroup.

In summary, the co-maximal subgroups can be expressed mathematically as a set of subgroups $\{M_1, M_2, \ldots, M_n\}$ in the group G such that each M_i is maximal, and the intersection of any pair of distinct subgroups is not contained in any other subgroup in the set.

Definition 2.3. Let G be a group, and H_1 and H_2 be co-maximal subgroups of G. The co-maximal subgroup graph $\Gamma(G)$ is a graph defined as follows:

Vertex Set: The vertex set of $\Gamma(G)$ consists of the co-maximal subgroups H_1 and H_2, denoted as $V(\Gamma(G)) = \{H_1, H_2\}$.

Edge Set: The edge set $E(\Gamma(G))$ consists of edges representing the relationships between co-maximal subgroups. Specifically, there is an edge between H_1 and H_2 if and only if $H_1 \cap H_2 = \{e\}$, where e is the identity element of G. Mathematically, $E(\Gamma(G)) = \{(H_1, H_2) \mid H_1 \cap H_2 = \{e\}\}$.

Graph Notation: The co-maximal subgroup graph $\Gamma(G)$ is then defined as the graph $(\{H_1, H_2\}, E(\Gamma(G)))$.

This mathematical definition encapsulates the essential properties of the co-maximal subgroup graph, where vertices represent co-maximal subgroups, and edges indicate that the corresponding subgroups have a trivial intersection.

Proposition 2.3. For any integer $n > 1$, there exist co-maximal subgroups in Z_n.

Proof.

To prove the proposition that for any integer $n > 1$, there exist co-maximal subgroups in Z_n, we need to show that there exist subgroups H and K in Z_n such that $H \cap K = \{0\}$.

Let's consider the subgroups H and K as follows:

1. $H = \{k \in Z_n \mid \gcd(k, n) = 1\}$: This subgroup consists of all integers in Z_n that are relatively prime to n.
2. $K = \{0\}$: This subgroup consists only of the additive identity element.

   Now, let's verify that H and K are co-maximal:

- Intersection: We need to show that $H \cap K = \{0\}$.

  $H \cap K = \{k \in Z_n \mid \gcd(k, n) = 1\} \cap \{0\} = \{0\}$

  This shows that the intersection is indeed the identity subgroup.

- Co-maximality: H and K are co-maximal if $\langle H, K\rangle = Z_n$, where $\langle H, K\rangle$ denotes the subgroup generated by H and K.

Notice that $\langle H, K\rangle$ includes all elements of H and K, but since $K = \{0\}$, $\langle H, K\rangle$ is essentially H. Therefore, $\langle H, K\rangle = H$.
Since H is the subgroup of all integers relatively prime to n, and Z_n consists of all integers modulo n, we can conclude that $\langle H, K\rangle = H = Z_n$.
Therefore, we have successfully shown that H and K are co-maximal subgroups in Z_n for any integer $n > 1$.

Proposition 2.4. If a and b are generators of co-maximal subgroups H_1 and H_2 in Z_n, then $\gcd(a, n) = \gcd(b, n) = 1$
Proof.
1. Generator of H_1: The subgroup H_1 generated by a is given by:

$H_1 = \langle a\rangle = \{a^k | k \in Z\}$
The generator a has the property that $\langle a\rangle \cap \{0\} = \{0\}$ (the identity element), making it a co-maximal subgroup.

2. Generator of H_2: Similarly, the subgroup H_2 generated by b is given by:

$H_2 = \langle b\rangle = \{b_k | k \in Z\}$
The generator b also has the property that $\langle b\rangle \cap \{0\} = \{0\}$, making it a co-maximal subgroup.
Now, let's prove that gcd(a, n) = gcd(b, n) = 1:

1. Assume $\gcd(a, n) > 1$**:** If $\gcd(a, n) > 1$, then there exists a prime factor p such that p divides both a and n. Let $a = p \cdot a'$ and $n = p \cdot n'$.

Now, consider H_1 generated by a. Since a is in H_1, all powers of a are also in H_1, including $p \cdot a'$. This implies that $p \cdot a'$ is in H_1, which means H_1 is not co-maximal with $\{0\}$, leading to a contradiction.
Therefore, $\gcd(a, n) = 1$.

2. Similar Argument for b: By a similar argument, we can show that $\gcd(b,n)=1$.

Hence, the proposition is proven: If a and b are generators of co-maximal subgroups H_1 and H_2 in Z_n, then $\gcd(a, n) = \gcd(b, n) = 1$

Theorem 2.5. The co-maximal subgroup graph in Z_n is connected, reflecting the interplay between elements that are relatively prime to each other.
Proof.
Let H_1 and H_2 be two arbitrary co-maximal subgroups in Z_n. By definition, these subgroups are generated by elements a and b respectively, such that *gcd(a, n) = gcd(b, n) = 1*.
Since gcd(a, n) = 1, a has a multiplicative inverse a^{-1} mod n. Similarly, since gcd(b, n) = 1, b has a multiplicative inverse b^{-1} mod n.
Now, consider the subgroup H_1 generated by a and the subgroup H_2 generated by b. We can define a path between H_1 and H_2 as follows:

1. Path from H_1 to H_2**:** Consider the sequence of subgroups H_1, $H_1 \cdot b$, $H_1 \cdot b_2$, …, $H_1 \cdot b^{k-1}$, $H_1 \cdot b^k = H_2$. Here, k is chosen such that $H_1 \cdot b_k = H_2$.

Notice that $H_1 \cdot b^i$ represents the subgroup generated by $a \cdot b^i$, and since gcd(a,n)=gcd(b,n)=1, all elements $a \cdot b^i$ are relatively prime to n. Therefore, each $H_1 \cdot b^i$ is a co-maximal subgroup.
The final subgroup in the sequence is H_2, and thus, a path exists from H_1 to H_2.

2. Reversibility of the Path**:** The path defined above is reversible by considering the sequence $H_2, H_2 \cdot a^{-1}, H_2 \cdot (a^{-1})^2, \ldots, H_2 \cdot (a^{-1})^k, H_1$. Here, k is chosen such that $H_2 \cdot (a^{-1})^k = H_1$. The reversibility of the path ensures that a path also exists from H_2 to H_1.

Therefore, a path exists between any pair of co-maximal subgroups in Z_n, and the co-maximal subgroup graph is connected. This connectivity reflects the interplay between elements that are relatively prime to each other in the group Z_n.

3. CENTRAL IDEA

Constructing a graph of co-maximal subgroups in the integer modulo Z_n group involves visually representing the relationships between subgroups. Here is a general guide to creating such a graph:
Graph Construction:

1. *Vertex Set:*
   - Each vertex represents a co-maximal subgroup of Z_n.
   - Enumerate the co-maximal subgroups, denoted as $H_1, H_2, \ldots, H_k$.
2. *Edge Set:*
   - Connect vertices with edges to represent co-maximal relationships.
   - There is an edge between H_i and H_j if and only if $H_i \cap H_j = \{0\}$.
3. *Visualization:*
   - Arrange vertices and edges to create a clear and insightful graph.
   - Consider using different colors or shapes to distinguish between vertices.

Example 3.1.
Let's consider an example where $n = 8$.

1. Co-Maximal Subgroups:
   - Subgroups of Z_8 include {0, 2, 4, 6} and {0, 4}.
   - These subgroups are co-maximal because their intersection is {0}.
2. Graph Representation:
   - Vertices: $H_1 = \{0,2,4,6\}$, $H_2 = \{0,4\}$.
   - Edge: Connect H_1 and H_2 since $H_1 \cap H_2 = \{0\}$.
3. Visualization:

```
H1        H2
|          |
|__________|
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1. Patterns:
   - Observe patterns in the graph, such as cyclic structures or hierarchical relationships.
   - Note any common factors in the generators of co-maximal subgroups.
2. Connectivity:
   - Verify that the graph is connected, as every non-trivial subgroup is co-maximal with its complement.

3. Number of Co-Maximal Subgroups:
    - Determine the number of co-maximal subgroups by finding pairs with trivial intersection.

Creating a graph of co-maximal subgroups provides a visual representation of relationships within the integer modulo Z_n group. Analyzing the graph can offer insights into the algebraic structure and modular arithmetic properties of the group. Remember to adapt the example and the analysis based on the specific value of *n* in your context.

CONCLUSION

The construction and analysis of a graph of co-maximal subgroups in the integer modulo Z_n group offer valuable insights into the algebraic structure and relationships within the group. This visual representation provides a clear and concise way to study the interplay between elements that are relatively prime to each other. Here, we provide a general guide to creating such a graph.

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