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APPLICATION OF EXTENDED COMPLEX PLANE AND RIEMANN SPHERE

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ABSTRACT: *In this article, description is made of the application of extended complex plane and stereographic projection is used to bijectively map points of the extended complex plane to the Riemann Sphere of a unit diameter. The relationship between the north pole of the Riemann sphere (N) and the point at infinity of the extended complex plane is well established. The distance between the north pole (N) and a point P in the plane is derived mathematically and we stated general applications of extended complex plane. Some striking figures which aided our step-by-step explanations were also presented.*

KEYWORDS: complex plane, Riemann sphere, infinity, stereographic projection, complex analysis, unit diameter.

Introduction

The extended complex plane is equipped with complex number $\mathbb{C}$ together with ∞ denoted $\mathbb{C} \cup \{\infty\}$ and it has only ∞ for $\mathbb{C}$ because the complex number $\mathbb{C}$ does not have natural ordering like set of real number exhibits. The extended complex plane is also known as compactified (closed) complex plane [4].

In an earlier paper [1], we gave a concrete picture of the extended complex plane and described how mapping of points of the extended complex plane and the Riemann Sphere is done using stereographic projection. We paid special attention to images of sums of complex numbers on the plane and on the sphere. Our goal in this paper is to do something similar in a different way and mention some applications of the theory. In [1], the complex plane was presented as intersecting the sphere of unit radius through its equator so that the centre of the equator of the sphere coincided with the origin of the complex plane. In this present work, the south-pole S of the sphere sits on the origin of the complex plane π and then stereographic projection is used to map points of the plane and sphere. The sphere here is of unit diameter.

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METHODOLOGY

Riemann Sphere and Stereographic projection.

In theory of geometry, stereographic projection is a bijective and smooth mapping that projects a sphere onto a plane [2]. The mapping is also conformal that is, it preserves angles at point of intersection of the curve but the mapping is not isomeric and does not preserve distances and area of figures [3].

Consider the sphere Σ with center at $(\xi, \eta, \zeta) = (0, 0, \frac{1}{2})$ and radius $r = \frac{1}{2}$, which intersects the origin of the complex plane at its south pole S. At the point S: (0, 0, 0) , the sphere is tangent to plane π whose equation is $\zeta = 0$. The point N: (0, 0, 1) is called the north pole [5].

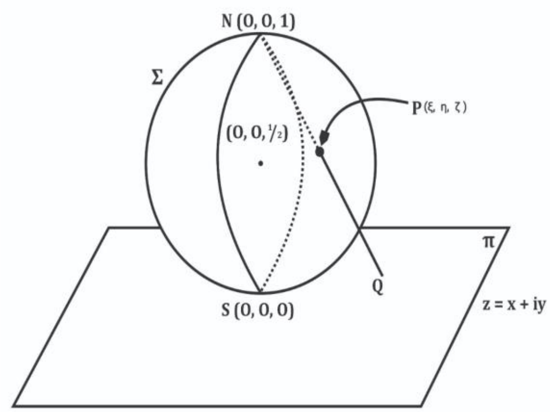

Figure (1)

The above sphere is often called the Riemann sphere

Suppose $P(\xi, \eta, \zeta)$ is a point on the sphere other than N : (0, 0, 1). A straight line joining N to P bisects the plane π at the point Q . We note that if π is a complex plane, then Q is a complex number $z = x + iy$ and if Q is a complex number, then the line QN bisects the sphere at exactly one point $P(\xi, \eta, \zeta)$.

Stereographic projection T establishes one –to- one mapping of a sphere Σ (without N) onto the complex plane : $\Sigma - \text{N} \rightarrow \pi$

Let S be sphere with radius $\frac{1}{2}$ and center at $(0, 0, \frac{1}{2})$, and let (ξ, η, ζ), be a point on this sphere.

It can be shown that $\xi^2 + \eta^2 = \zeta(1 - \zeta)$

The most general equation of a sphere with coordinate (a, b, c) is

$$(x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 = r^2 \qquad (2.1)$$

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Where $(x_1,\ x_2,\ x_3)$ is a point on the sphere.

In this case, (a, b, c) = (0, 0, $\frac{1}{2}$)

And r = $\frac{1}{2}$, $(\ \xi,\ \eta,\ \zeta\ ) = (x_1,\ x_2,\ x_3)$

$$\xi^2 + \eta^2 + (\zeta - \tfrac{1}{2})^2 = (\tfrac{1}{2})^2 \qquad (2.2)$$

$$\xi^2 + \eta^2 = \zeta(1-\zeta) \qquad (2.3)$$

Now we want to find the formula for the stereographic projection $z = x + iy$ of the point $(\ \xi,\ \eta,\ \zeta\ )$, on *R*. We will find equations expressing $\xi,\ \eta$ and ζ in terms of *x* and *y*.

Line QN bisects Σ at $(\xi,\ \eta,\ \zeta\ )$. Since N: (0, 0, 1), the equation of QN are given by

$$x' = 0 + \xi t \qquad (2.4)$$

$$y' = 0 + t\eta \qquad (2.5)$$

$$z' = 1 + (\zeta - 1)t \qquad (2.6)$$

Setting $z' = 0$ yileds

$$t = \frac{1}{1-\zeta} \qquad (2.7)$$

Hence $Z = x + yi = \xi t + it\eta = \frac{\xi + i\eta}{1-\zeta}$ (2. 8)

Equation (2.8) can be written as

$$\xi + i\eta = z(1-\zeta) \qquad (2.9)$$

But $\xi,\ \eta, and\ \zeta$ are real,

$$\xi - i\eta = \bar{z}(1-\zeta) \qquad (2.10)$$

Since $(\xi,\ \eta,\ \zeta\ )$ is a point on the sphere, from equation (2.3) we have

$$\xi^2 + \eta^2 = \zeta(1-\zeta) \qquad (2.11)$$

Multiplying equation (2.9) by (2.10) we have

$$\xi^2 + \eta^2 = z\bar{z}(1-\zeta)^2 \qquad (2.12)$$

Applying equation (2.11)

$\zeta = z\bar{z}(1-\zeta)$ solving for ζ

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$\zeta = \frac{z\,\bar{z}}{z\,\bar{z} + 1}$ (2. 13)

Adding (2.9) and (2.10), we have

$\xi = \frac{z(1-\zeta) + \bar{z}(1-\zeta)}{2} = \frac{z+\bar{z}}{2(z\bar{z}+1)}$ (2. 14)

Subtracting (2.9) from (2.10), we have

$\eta = \frac{z+\bar{z}}{2i(z\bar{z}+1)}$ (2.15)

In terms of $z = x + yi,$ we represent

$(\xi,\ \eta,\ \zeta) = \left(\frac{x}{x^2+y^2+1}, \frac{y}{x^2+y^2+1}, \frac{x^2+y^2}{x^2+y^2+1}\right)$ (2.16)

RESULTS AND DISCUSSIONS

Computation of Length NP

Consider the figure blew

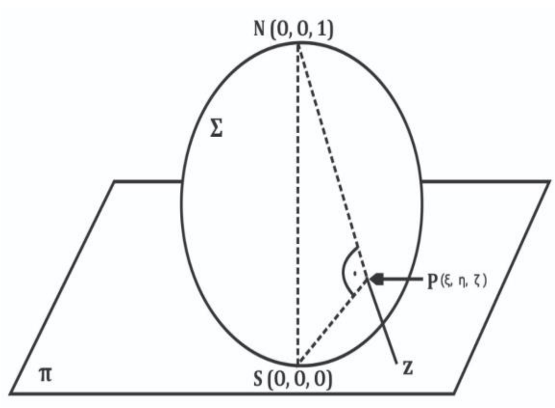

Figure (2)

Since P is a point on the sphere, we have

$(NS)^2 = (NP)^2 + (PS)^2$ (3.1)

But NS = $\frac{1}{2} + \frac{1}{2} = 1$ (3. 2)

SP = $\sqrt{\xi^2 + \eta^2 + \zeta^2}$ (3.3)

Hence

$1 = \xi^2 + \eta^2 + \zeta^2 + (NP)^2$ (3. 4)

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Since P is located on the sphere,

$$\xi^2 + \eta^2 = \zeta(1 - \zeta) \qquad (3.5)$$

Therefore

$$1 = \zeta(1 - \zeta) + \zeta^2 + (NP)^2 \qquad (3.6)$$

Simplify (3.6) we have

$$NP = \sqrt{1 - \zeta} \qquad (3.7)$$

In terms of x and y, (3.7) can be written as

$$NP = \frac{1}{\sqrt{x^2+y^2+1}} \qquad (3.8)$$

NP is the distance on the sphere that corresponds to the distance between any z and infinity on the plane. The distance between zero and infinity on the plane will correspond to 1, using (3.8). This is easy to see as zero corresponds to S, the south-pole and infinity corresponds to N, the north-pole and the distance between the north pole and the south pole is 1 as the sphere is of unit diameter.

Relationship between the north pole of the Riemann sphere (N) and the point at infinity of the extended complex plane

The complex sequence (z_n) approaches infinity

$$\lim_{n\to\infty} z_n = \infty \qquad (3.9)$$

If for any given $M > 0$, there exists $N \in N'$ such that for $n > N$,

$$|z_n| > M \qquad (3.10)$$

Let (z_n) be a complex sequence approaching infinity. The corresponding sequence of points on the sphere is (p_n).

As $z_n \to \infty$, p_n approaches the poles N

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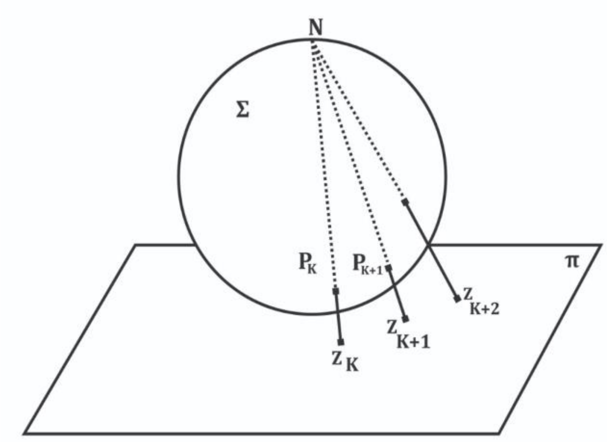

Figure (3)

We say that N corresponds to the point at infinity of the complex plane.

A complex plane equipped with the point at infinity is called extended called plane. The sphere Σ (with N) is called the Riemann sphere, the mapping of the Riemann sphere onto the extended complex plane is called the stereographic projection.

Note , the stereographic projection establishes an injective mapping between the extended complex plane and the Riemann sphere.

Application of Extended Complex plane and the Riemann sphere.

1. It facilitates the use of infinity for complex numbers including all family of meromorphic functions mapping to the Riemann sphere. E.g For any complex number Z, we have, $z + \infty = \infty$, $z \times \infty = \infty$, $\frac{z}{0} = \infty$, $\frac{z}{\infty} = 0$, $\infty \times \infty = \infty$, $\infty + \infty = \infty$ $\frac{\infty}{0} = \infty$, $\frac{0}{\infty} = 0$, while $\frac{0}{0}$, $\infty - \infty$ and $\frac{\infty}{\infty}$ are left undefined provided $z \neq 0$
2. It aids visualization of lines and planes as points in the disk and for visualization of directional data in crystallography and geology.
3. In arithmetic geometry, stereographic projection from the unit circle can be used to describe all primitive Pythagorean triple.
4. In photography, most fisheye lenses use a stereographic projection to capture images from a wide- angle view
5. In quantum mechanics, points on the complex projective line serve as natural values for photon polarization states , spin states of massive particles of spine $\frac{1}{2}$, and entire 2- state particles

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CONCLUSION

The mapping of points between the sphere of unit diameter and the extended complex plane is clearly explained using stereographic projection. The intersection of the sphere and plane at the origin of the plane and the south pole of the sphere allowed us to easily derive the mapping equations as a straight line drawn from the North pole will touch the sphere in exactly one point before going to touch the plane. The points where the straight line touch the line and the sphere is the points that corresponds to each other. The equations (2.8) and (2.16) map points of the sphere to the plane and those of the plane to the sphere respectively. Clearly infinity is mapped to the North pole and there exists a one to one correspondence between the points of the sphere and plane.

Mention was made of application of the theory in various areas and we also derived equation that map the distance between any point on the plane and infinity to its corresponding distance on the sphere.

Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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