International Journal of Mathematics and Statistics Studies
Vol.12, No.4, pp.52-61, 2023
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)
Website: https://www.eajournals.org/
Publication of the European Centre for Research Training and Development –UK

On Some Properties of a Class of Analytic Functions Defined by Opoola Differential Operator

R. A. Bello
Department Mathematics and Statistics, Kwara State University, Malete, Kwara State, Nigeria.
risikat.bello@kwasu.edu.ng

doi: https://doi.org/10.37745/ijmss.13/vol12n45261 Published July 21 2024

Citation: R. A. Bello (2023) On Some Properties of a Class of Analytic Functions Defined by OpoolaDifferential Operaor, *International Journal of Mathematics and Statistics Studies*, Vol.12, No.4, pp **52**-61

ABSTRACT: The Fekete-Szego functional upper bounds and the Second Hankel Determinant upper bounds for a class of analytical functions defined by the Opoola Differential Operator are found in this study. The estimations made are accurate with thetheorems proved.

KEYWORDS: Analytic functions, Opoola Differential Operator, Starlike functions, Univalent functions, Coefficient bounds, Fekete-Szego functional, Second Hankel Determinant and Subordination.

INTRODUCTION

Let A denote the class of function $f(z)$ analytic in the open unit disk $U = \{z \in C: |z| < 1\}$ and let $S \in A$ denote the class of analytic functions $f(z)$ in U which are univalent in U and of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (1)$$

A function f(z) belonging to the class S is called starlike function if f(z) maps the unit disk U onto a starlike domain. A necessary and sufficient conditions for a f(z)∈ S to be starlike with respect to the origin is that;

$$Re\left\{\frac{f^{F}(z)}{f(z)}\right\}, > 0, (z \in u) \qquad (2)$$

The class of starlike functions is denoted as S^*

A function f(z) $\in S$ is said to be starlike of order α , $0 \leq \alpha < 1$, If

$$Re\left\{\frac{f^{F}(z)}{f(z)}\right\}, > \alpha , (z \in u) \qquad (3)$$

The class of starlike functions of order α is denoted as $S^*(\alpha)$

A function $f(z) \in S$ is called a Convex function if $f(z)$ maps the unit dist U onto a convex domain. A function $f(z)$ is said to be a Convex function if and only if

$$\Re\left\{1+\frac{zf''(z)}{f\prime(z)}\right\}>0 \qquad (z\in U)) \tag{4}$$

The class of convex functions is denoted as K.
A function $f(z) \in S$ is said to be Convex of order α, $0 \leq \alpha < 1$ if

$$Re\left\{1+\frac{zf''(z)}{f\prime(z)}\right\}>\alpha \qquad (z\in U)) \tag{5}$$

The class of all Convex functions of order α is denoted as $K(\alpha)$.

Definition 1.1. *Let f and g be analytic in U. Then $f(z)$ is subordinate to $g(z)$ denoted by $f \prec g$ if there exist an analytic function $\omega(z)$ with $\omega(0)=0$, $|\omega(z)|<1$ such that $f(z)=g(\omega(z))$. In particular if $g(z)$ is univalent in U then $f \prec g \Longleftrightarrow f(0)=g(0)$ and $f(U) \subset g(U)$.*

The q^{th} Hankel determinant $H_q(n)$,$q \geq 1$,$n \geq 1$ for a function $f(z) \in A$ and have the form (1)is defined as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \vdots & & & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}$$

In recent years, attention has been given to finding estimates of the determinant $H_q(n)$. The Fekete-Szego functional $|a_3-\lambda a_2^2|$ is $H_2(1)$.For $f(z) \in S$ it is known that $H_2(1) \leq 1$. see [1], The Second Hankel Determinant $H_2(2)=|a_2a_4-a_3^2|$ has received more attention from many authors.The sharp upper bound of $H_2(2)$ for starlike and convex functions was studied in [2] and the authors obtained $H_2(2) \leq 1$ and $H_2(2) \leq \frac{1}{8}$ respectively. Many other results have been obtained for $H_2(2)$ for a variety of subclass of S,most of which are subclass of S^*.
Noticing that several subclass of univalent functions are characterized by the quantities $\frac{zf'(z)}{f(z)}$ or $\left\{1+\frac{zf''(z)}{f\prime(z)}\right\}$ lying in a region in the right half plane, Ma and Minda [3] considered the classes.

$$ST(\phi)=\left\{f\in A:\frac{zf'(z)}{f(z)}\right\} \tag{6}$$

$$CV(\phi)=\left\{f\in A:\frac{1+zf''(z)}{f\prime(z)}\right\} \tag{7}$$

For $\phi(z) = \frac{1+Az}{1+Bz}, \quad -1 \leq B < A < 1,$
The class $ST(\phi)$ reduces to the familiar class consisting of Janowski starlike functions denoted by $ST(A, B)$.The corresponding class of convex functions is denoted by $CV(A, B)$.
The special case of $A = 1 - 2\alpha, \quad B = -1, \quad 0 \leq \alpha < 1$,$ST(\phi)$ and $CV(\phi)$gives the classes of starlike functions of orderα and convex functions of order α respectively. Let SC be the class of functions $f \in S$ with the quantity $\frac{zf'(z)}{f(z)}$ lying in the region bounded by the cardiod given by $(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0$ Thus a function $f(z) \in SC$ if $\frac{zf'(z)}{f(z)} \in CAR$ where,

$$CAR = \omega \in \mathbb{C} : \omega = 1 + \frac{4}{3}z + \frac{2}{3}z^2, (|z| < 1) \tag{8}$$

The class SC was investigated by Kanika [4]
We say that $f \in S$ belongs to the class S_nC if

$$\frac{D^{n+1}_{\mu,\beta,t}f(z)}{D^{n}_{\mu,\beta,t}f(z)} \prec \phi(z) \tag{9}$$

where $D^n_{\mu,\beta,t}$ is the Opoola Differential Operator, $n \in N \cup 0$ and

$$\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2 \quad z \in U \tag{10}$$

Remark[1]: The class $S_nC(\mu, \beta, t)$ reduces to the class S_nC when $\beta = \mu$and $t = 1$

Remark[2]: The class $S_nC(\mu, \beta, t)$ reduces to the class SC when $\beta = \mu, t = 1$and $n = 0$.

In this paper, we obtain the initial coefficient estimates a_2, a_3 and a_4 for functions belonging to the class S_nC.The Upper bounds for the Fekete-Szego functional and the second Hankel Determinant for functions belonging to the class S_nC are also established. Furthermore,when $n = 0$ and $\phi(z) = \sqrt{(1 + z^2)} + z$, the class S_nC becomes e class $S^*(q)$ sudied in [1]

2 Main Result

Let Ω be the class of analytic functions of the form

$$\omega(z) = c_1z + c_2z^2 + c_3z^3 + \ldots \tag{11}$$

such that $|c_k| \leq 1, \quad k = 1, 2, 3, \ldots$

Lemma 2.1

If $\omega \in \Omega$, then for any $t \in \mathbb{R}$

$$|c_2 - \lambda c_1^2| \left\{ \begin{array}{ccc} -t & if & < -1 \\ 1 & if & -1 \leq \lambda \leq 1 \\ t & if & \lambda > 1 \end{array} \right\}$$

Lemma 2.2

If $\omega \in \Omega$, for any complex number λ

$|c_2 - \lambda c_1^2| \leq max\{1 : |t|\}$

The result is sharp for $\omega(z) = z^2$ or $\omega(z) = z$

Lemma 2.3

If $\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + .\therefore \in \Omega$,

then $|c_1^3 + c_3 + 2c_1 c_2| \leq 1$

Theorem 3.1

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S_n C(\mu, \beta, t)$ then

$$|a_2| \leq \frac{4}{3 \cdot [1 + (1 + \beta - \mu)t]^n}, \qquad |a_3| \leq \frac{11}{3^2 \cdot [1 + (2 + \beta - \mu)t]^n}, \qquad |a_4| \leq \frac{68}{3^3 \cdot 3 \cdot [1 + 3 + (\beta - \mu)t]^n}$$

Proof:

Since $f \in S_n C(\mu, \beta.t)$,

We have that,

$$\frac{D_{\mu,\beta,t}^{n+1} f(z)}{D_{\mu,\beta,t}^{n} f(z)} = \phi(\omega(z))$$

where $\phi(z)$ is given as (1),

Thus

$$\frac{D_{\mu,\beta,t}^{n+1} f(z)}{D_{\mu,\beta,t}^{n} f(z)} = 1 + \frac{4}{3}\omega(z) + \frac{2}{3}(\omega(z))^2 \qquad (12)$$

Let $\omega = c_1 z + c_2 z^2 + c_3 z^3 + ... \in \Omega$

Thus from (12) we obtain

$$D_{\mu,\beta,t}^{n+1} f(z) = D_{\mu,\beta,t}^{n} f(z)[1 + \frac{4}{3}\omega(z) + \frac{2}{3}(\omega(z))^2]$$

Therefore,

$$D^{n+1}_{\mu,\beta,t}f(z) = z+[1+(1+\beta-\mu)t]^{n+1}a_2z^2+[1+(2+\beta-\mu)t]^{n+1}a_3z^3+[1+(3+\beta-\mu)t]^{n+1}a_4z^4+... \tag{13}$$

and

$$D^{n}_{\mu,\beta,t}f(z)[1+\frac{4}{3}\omega(z)+\frac{2}{3}(\omega(z))^2]$$
$$= z+[\frac{4}{3}c_1+[1+(1+\beta-\mu)t]^n a_2]z^2+[\frac{4}{3}c_2+\frac{2}{3}c_1^2+[1+(1+\beta-\mu)t]^n a_2\frac{4}{3}c_1+[1+(2+\beta-\mu)t]^n a_3]z^3$$
$$...+\frac{4}{3}c_1c_2+[1+(1+\beta-\mu)t]a_2\frac{4}{3}c_2+\frac{2}{3}[1+(1+\beta-\mu)t]^n a_2c_1^2+[1+(2+\beta-\mu)t]^n a_3+\frac{4}{3}c_1$$
$$+[3+(1+\beta-\mu)t]^n a_4]z^4$$

(14)

Equating and Comparing the coefficients in (13)and (14) we have

$$[1+(1+\beta-\mu)t]^{n+1}a_2 = \frac{4}{3}c_1+[1+(1+\beta-\mu)t]^n a_2$$

i.e

$$a_2 = \frac{4}{3\cdot[1+(1+\beta-\mu)t]^n}c_1$$

$$[1+(2+\beta-\mu)t]^{n+1}a_3 = \frac{4}{3}c_2+\frac{2}{3}c_1^2+\frac{4}{3}[1+(1+\beta-\mu)t]^n a_2c_1+[1+(2+\beta-\mu)t]^n a_3 \tag{15}$$

which gives

$$a_3 = \frac{4}{3\cdot 2\cdot[1+(2+\beta-\mu)t]^n}c_2+\frac{11}{3^2\cdot[1+(2+\beta-\mu)t]^n}c_1^2 \tag{16}$$

and

$$[1+(3+\beta-\mu)t]^{n+1}a_4 = \frac{4}{3}c_3+\frac{4}{3}c_1c_2+\frac{4}{3}[1+(1+\beta-\mu)t]^n a_2c_2+$$
$$\frac{2}{3}[1+(1+\beta-\mu)t]^n a_2c_1^2+\frac{4}{3}[1+(2+\beta-\mu)t]^n a_3c_1+[1+(3+\beta-\mu)t]^n a_4$$

Therefore,

$$a_4 = \frac{4}{3\cdot 3\cdot [1+(3+\beta-\mu)t]^n}c_3 + \frac{36}{3^2\cdot 3\cdot [1+(3+\beta-\mu)t]^n}c_1c_2 + \frac{68}{3^3\cdot 3\cdot [1+(3+\beta-\mu)t]^n}c_1^3 \tag{17}$$

From equation (15)and using $|c_k| \leq 1$ we obtain

$$\begin{aligned}
|a_3| &= \left|\frac{4}{3\cdot 2\cdot [1+(2+\beta-\mu)t]^n}\right| \left|c_2 + \frac{22}{3^2\cdot 2\cdot [1+(2+\beta-\mu)t]^n}\cdot\frac{3\cdot 2\cdot [1+(2+\beta-\mu)t]^n c_1^2}{4}\right| \\
&= \frac{4}{3\cdot 2\cdot [1+(2+\beta-\mu)t]^n}\left|\left(c_2 + \frac{22}{3^2\cdot 2\cdot [1+(2+\beta-\mu)t]^n}\cdot\frac{3\cdot 2\cdot [1+(2+\beta-\mu)t]^n c_1^2}{4}\right)\right| \\
&= \frac{4}{3\cdot 2\cdot [1+(2+\beta-\mu)t]^n}\left|c_2 + \frac{22}{12}c_1^2\right| \\
&= \frac{4}{3\cdot 2\cdot [1+(2+\beta-\mu)t]^n}\left|c_2 - (-\frac{22}{12})c_1^2\right|
\end{aligned}$$

Applying Lemma (2.1),

$$|a_3| \leq \frac{4}{3\cdot 2\cdot [1+(2+\beta-\mu)t]^n}\left[(-\frac{22}{12})\right] = \frac{11}{9\cdot 3^n}$$

$$|a_3| \leq \frac{11}{9\cdot [1+(2+\beta-\mu)t]^n}$$

Also from equation (17) we have

$$\begin{aligned}
a_4 &= \frac{4}{3\cdot 3\cdot [1+(3+\beta-\mu)t]^n}c_1c_2 + \frac{68}{81\cdot [1+(3+\beta-\mu)t]^n}c_1^3 + \frac{4}{9\cdot [1+(3+\beta-\mu)t]^n}c_3 \\
&= \frac{68}{3^3\cdot 3\cdot [1+(3+\beta-\mu)t]^n}\left(c_1^3 + \frac{36}{68}c_3 + \frac{27}{17}c_1c_2\right) \\
&\leq \frac{68}{3^3\cdot 3\cdot [1+(3+\beta-\mu)t]^n}\left(c_1^3 + 2c_1c_2 + c_3\right)
\end{aligned}$$

and lemma (2.3)we obtain

$$\begin{aligned}
|a_4| &\leq \frac{68}{3^3\cdot 3\cdot [1+(3+\beta-\mu)t]^n}|c_1^3 + c_3 + 2c_1c_2| \\
&\leq \frac{68}{81\cdot [1+(3+\beta-\mu)t]^n}
\end{aligned}$$

which complete the proof.

Theorem 3.2

Let $\sigma_1 = \frac{5\cdot 2^{2n}}{16\cdot 3^{n},}$ $\sigma_2 = \frac{17\cdot 2^{2n}}{16\cdot 3^{n}}$, If $f(z) \in S_nC(\mu.\beta,t)$, then for any real number λ

$$|a_3 - \lambda a_2^2| \leq \left\{ \begin{array}{rl} \frac{1}{9}\left(\frac{11}{[1+(2+\beta-\mu)t]^n} - \frac{16}{[1+(1+\beta-\mu)t]^{2n}}\lambda\right) & if \quad \lambda < \sigma_1 \\ \frac{4}{6\cdot[1+(2+\beta-\mu)t]^n} & if \quad \sigma_1 \leq \lambda \leq \sigma_2 \\ -\frac{1}{9}\left(\frac{11}{[1+(2+\beta-\mu)t]^n} - \frac{16}{[1+(1+\beta-\mu)t]^{2n}}\lambda\right) & if \quad \lambda > \sigma_2 \end{array} \right\} \quad (19)$$

Proof:
If $f(z) \in S_nC(\mu.\beta,t)$,then from equation(15)and (16) we get

$$a_3 - \lambda a_2^2 = \frac{4}{6\cdot[1+(2+\beta-\mu)t]^n}c_2 + \frac{22}{18\cdot[1+(2+\beta-\mu)t]^n}c_1^2 - \lambda\frac{16}{9\cdot[1+(1+\beta-\mu)t]^{2n}}c_1^2$$

$$= \frac{4}{6\cdot[1+(2+\beta-\mu)t]^n}\left[c_2 - \left(\frac{8\cdot[1+(2+\beta-\mu)t]^n}{3\cdot[1+(1+\beta-\mu)t]^{2n}}\lambda - \frac{11}{6}\right)c_1^2\right] \quad (20)$$

By applying lemma (2.1),equation (20) yields

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6\cdot[1+(1+\beta-\mu)t]^n}\left(\frac{11}{6} - \frac{8\cdot[1+(2+\beta-\mu)t]^n}{3\cdot[1+(1+\beta-\mu)t]^{2n}}\lambda\right)$$

For

$$\frac{8\cdot[1+(2+\beta-\mu)t]^n}{3\cdot[1+(1+\beta-\mu)t]^{2n}}\lambda - \frac{11}{6} < -1$$

i.e

$$\lambda < \frac{5\cdot[1+(1+\beta-\mu)t]^{2n}}{16\cdot[1+(2+\beta-\mu)t]^n}$$

and taking$\sigma_1 = \frac{5\cdot[1+(1+\beta-\mu)t]^{2n}}{16\cdot[1+(2+\beta-\mu)t]^n}$
we obtain that

$$|a_3 - \lambda a_2^2| \leq \frac{1}{9}\left(\frac{11}{[1+(2+\beta-\mu)t]^n} - \frac{16}{[1+(1+\beta-\mu)t]^{2n}}\lambda\right) \quad (21)$$

when $\lambda < \sigma_1$
Also, using Lemma 2.1, inequality (20) yields

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6\cdot[1+(2+\beta-\mu)t]^n}$$

for

$$-1 \leq \frac{8 \cdot [1+(2+\beta-\mu)t]^n}{3 \cdot [1+(1+\beta-\mu)t]^{2n}}\lambda - \frac{11}{6} \leq 1$$

that is, for

$$\frac{5 \cdot [1+(1+\beta-\mu)t]^{2n}}{16 \cdot 3^n} \leq \lambda \leq \frac{17 \cdot 2^{2n}}{16 \cdot [1+(2+\beta-\mu)t]^n}$$

Taking $\sigma_2 = \frac{17 \cdot [1+(1+\beta-\mu)t]^{2n}}{16 \cdot [1+(2+\beta-\mu)t]^n}$
we obtain that

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6 \cdot [1+(2+\beta-\mu)t]^n} \quad \text{if} \quad \sigma_1 \leq \lambda \leq \sigma_2 \tag{22}$$

Applying Lemma (2.1) again to inequality (20) we have,

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6 \cdot [1+(2+\beta-\mu)t]^n} \left(\frac{8 \cdot [1+(2+\beta-\mu)t]^n}{3 \cdot [1+(1+\beta-\mu)t]^{2n}}\lambda - \frac{11}{6} \right)$$

for

$$\frac{8 \cdot [1+(2+\beta-\mu)t]^n}{3 \cdot [1+(1+\beta-\mu)t]^{2n}}\lambda - \frac{11}{6} > 1$$

that is,

$$|a_3 - \lambda a_2^2| \leq \frac{1}{9} \left(\frac{16}{[1+(1+\beta-\mu)t]^{2n}}^n \lambda - \frac{11}{[1+(2+\beta-\mu)t]^{2n}}^n \right) \tag{23}$$

for $\lambda > \sigma_2$
Combining inequality (21)(22)and (23)we obtain the result of the theorem.

Theorem 3.3

If $f(z) \in S_nC(\mu, \beta, t)$ then for any complex number λ

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6[1+(2+\beta-\mu)t]^{2n}}^n \max\{1; \left| \frac{11}{6} - \frac{8[1+(2+\beta-\mu)t]^{2n}}{3[1+(1+\beta-\mu)t]^{2n}} \right| \}$$

Proof:
From inequality (15)and(16)we have that

$$|a_3 - \lambda a_2^2| = \left| \frac{4}{6[1+(1+\beta-\mu)t]^n} \left[c_2 - \left(\frac{8[1+(1+\beta-\mu)t]^n \lambda}{3[1+(2+\beta-\mu)t]^{2n}} - \frac{11}{6} \right) c_1^2 \right] \right|$$

$$|a_3 - \lambda a_2^2| = \frac{4}{6[1+(1+\beta-\mu)t]^n}\left|c_2 - \left(\frac{8[1+(1+\beta-\mu)t]^n\lambda}{3[1+(2+\beta-\mu)t]^{2n}} - \frac{11}{6}\right)c_1^2\right|$$

By applying lemma 2.2 we have

$$|a_3 - \lambda a_2^2| \leq \frac{4}{6[1+(2+\beta-\mu)t]^n}\max\left|\{1; \frac{8[1+(2+\beta-\mu)t]^n}{3[1+(1+\beta-\mu)t]^{2n}} - \frac{11}{6}\right|\}$$

This completes the proof.

Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)
Website: https://www.eajournals.org/
Publication of the European Centre for Research Training and Development –UK

REFERENCES

Ma W. C and D. Minda,*A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Anal- ysis*(Tianjin 1992), Pp.157-169, Conf. Proc. Lecture Notes Anal., I Int. Press, Cambridge, M.A

Raina R. K. and Sokol. J, *On Coefficient Estimate For A Class of Starlike Functions*, 2000 Mathematics Subject Classification. Primary 30C45

Bello. R. A and Opoola. T.O, *Upper Bounds for Fekete-Szego functions and the Second Hankel Determinants for a class of Starlike*. IOSR. Journal of Mathematics.Vol.13, Issue 2,(March-April,2017) Pp 34-39 i E. Kedioglu, *On Subclass of univalent functions with negative coefficients*. Applied Mathematics and computation,146(2003)351-358

Fatunsi L. M Fatunsin, T. O. Opoola. (2017).*New results on subclasses of analytic functions defined by Opoola differential Operator.* Journal of Mathematics and System Sciencec,7.289-295.doi:10.17265/2159-5291/2017.10.003)

Fekete. M. Szego, *Einebemerkung uber ungerade functionen*. London Math.Soc.8 (1933)85-89.

Jantency. A., Halim,, S. and Dans. M., *Hankel Determinants for Starlike and Convex functions*. Int. Journal, Math.Analysis,1(2007), No 13,619-625

Kanika. S. *Geometric Properties of a class of starlike functions* Amer Journal of Mathematics and Mathematical Sciences.3(1), January-June 2014.ISSN 2278-0874, pp 59-65

Kowalezyk. B. and Lecko. A. *The Fekete-Szego Inequality for close-to-convex functions with respect to a certain starlike function dependent on a real parameter* Journal of Inequalities and Applications,2014

A. O. Lasode, T. O. Opoola (2022d) *Some properties of a family of univalent functions defined by a generalized Opoola differential Operator*. General Mathematics,30(1),3-13

R.M.Ali and R.N. Seenivasagan, *Coefficient Bounds for P-Valent Func- tions*, Applied Mathematics and Computation,(2007)187, 35-46

S. K. Lee, V. Ravichandran and S. Shamani, *Coefficient Bounds*, Vol. 10.Iss.3, Art.71(2009).