International Journal of Mathematics and Statistics Studies, 12 (3), 1-16, 2024

Print ISSN: 2053-2229 (Print),

Online ISSN: 2053-2210 (Online)

Website: https://www.eajournals.org/

Publication of the European Centre for Research Training and Development -UK

Stochastic Calculus and Its Impact On Analyzing Option Pricing

[1]**Popoola Odunayo Emmanuel**
Department of Mathematical Sciences, Bamidele Olumilua University of Education Science and Technology, Ikere – Ekiti, Ekiti-State
ORCID: 0000-0002-4801-9574

[2]**Oluwabunmi Veronica Kehinde-Dada PhD.**
Department of Counselling Psychology, Bamidele Olumilua University of Education Science and Technology, Ikere – Ekiti, Ekiti-State
ORCID: 0000-0002-9143-4781

doi: https://doi.org/10.37745/ijmss.13/vol12n3116 Published April 2, 2024

Citation: Popoola O.E. and Kehinde-Dada O.V. (2024) Stochastic Calculus and Its Impact On Analyzing Option Pricing, *International Journal of Mathematics and Statistics Studies*, 12 (3), 1-16

ABSTRACT: *This research study the behaviour of the correction of Black-Scholes portfolios based on historical stock price data. We build a model that simulates one sample path of the stock price stochastic process at discrete time steps and track the correction over a time interval as it relates to the change in stock price over time. We also study the effect that relaxing this assumption has on the self-financing property of the replicating portfolio. We show that as the frequency of the time steps increases, the correction is more likely to be close to 0. We also show that the majority of historical stock return series that studied have caused the replicated portfolio to have a positive correction. We conclude that the Black-Scholes model can be used to find the no-arbitrage rational price for an option, a financial instrument that derives its value from the value of an underlying asset.*
KEYWORDS: stochastic calculus, Black-Scholes, stock price, financial mathematics, asset

INTRODUCTION

Before delving into stochastic calculus, it's beneficial to revisit the foundational concepts of traditional differential and integral calculus. The core of calculus lies a profound notion: the ability to determine a function's values based on its rate of change (Gregory, 2014). To illustrate, consider a scenario where $f(t)$ represents the position of a particle in one dimension at time t, and we possess information regarding the rate of change.

$$df(t) = c(t, f(t))dt \tag{1}$$

$$\frac{df}{dt} = f'(t) = c(t, f(t)) \quad (2)$$

At any given time t, the graph of t shifts by an infinitesimal increment along a straight line with a slope determined by $c\,(t, f\,(t))$. This exemplifies a differential equation, where the rate of change relies on both time and position. When provided with an initial condition such as $f\,(0) = x_0$, and subject to certain assumptions about the rate function c, the function can be defined and expressed as follows:

$$f(t) = x_0 + \int_0^t c(s, f(s))ds \quad (3)$$

Occasionally, it's feasible to perform integration and precisely calculate the function. However, when exact solutions evade us, leveraging computational methods becomes indispensable. One such straightforward approach is Euler's method, where we select a small increment Δt and express the process as follows:

$$f\,((k + 1)\Delta t) = f(k\,\Delta t) + \Delta tc(k\Delta t, f(k\Delta t)) \quad (4)$$

Stochastic calculus operates on a similar principle, but with the incorporation of randomness into the change. In essence, we aim to decipher equations like the following:

$$dX_t = m(t, X_t)dt + \sigma(t, X_t)\, dB_t \quad (5)$$

Here, Bt represents a standard Brownian motion. This equation exemplifies a stochastic differential equation (SDE). We interpret this equation as indicating that at time t, Xt evolves akin to a Brownian motion with a drift $m(t, Xt)$ and a variance of $[\sigma(t, Xt)]2$. One approach is the stochastic Euler method, which relies on Monte Carlo simulations of the process. As outlined by Gregory F. L (2014), the formula for this method is:

$$\sqrt{}X((k + 1)\Delta t) = X(k\Delta t) + \Delta tm(k\Delta t, X(k\Delta t)) + \Delta t\sigma\,(k\Delta t, X(k\Delta t))N_k. \quad (6)$$

where N_k is a $N(0,1)$ random variable.

In our formulation, our focus is primarily on delineating the stochastic integral. We define Xt as a solution to equation (1.1.1) if:

$$X_t = X_0 + \int_o^t m(s, X_s)ds + \int_o^t \sigma(s, X_s)dB_s \quad (7)$$

The ds integral aligns with the conventional integration from calculus; while the integrand $m(s,Xs)$ incorporates randomness, this doesn't pose any hurdles in defining the integral. Our primary challenge lies in ascribing precise significance to the second component, and more broadly, to:

International Journal of Mathematics and Statistics Studies, 12 (3), 1-16, 2024
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)
Website: https://www.eajournals.org/
Publication of the European Centre for Research Training and Development -UK

$$\int_o^t A_s dB_s \tag{8}$$

Among the array of methods available for stochastic integration, the one most prevalent in financial mathematics is the Itoˆ integral approach.

Over the past quarter-century, financial markets have undergone profound evolution. The advent of financial derivatives like options and futures on various underlying (stocks, bonds, currencies) has ushered in a new era of securitizing financial risks. The fundamental concept behind financial derivatives is to procure insurance for volatile assets through market participation, thereby seeking counterparts willing to share the risks and potential profits of uncertain future market developments. The pricing of these financial instruments hinges on sophisticated mathematical frameworks, notably Itoˆ stochastic calculus.

At the core of this mathematical theory lies the depiction of uncertain prices through Brownian motion and associated differential equations. The groundbreaking work of Black, Scholes, and Merton in 1973, particularly their Nobel Prize-winning Black-Scholes formula for pricing European call options, marked a pivotal moment in understanding and valuing financial derivatives. Their pioneering approach laid the groundwork for modern financial mathematics, leveraging advanced methodologies such as martingale theory and stochastic control to address the pricing challenges posed by an exponentially growing array of derivatives worldwide. Black Scholes, and Merton approached the task of pricing options with a perspective akin to that of physicists, employing rigorous mathematical reasoning to unlock insights into the complex dynamics of financial markets. They embarked on their journey by postulating a rational model for the pricing dynamics of a risky asset, a pursuit with a lengthy historical backdrop. Empirical investigations, rooted in statistical and econometric analyses, have revealed the limited predictability of future price changes by mathematical models. This observation is encapsulated in the economic literature as the "random walk hypothesis," suggesting that prices follow a random path over discrete, equidistant time intervals. In finance, however, the focus is predominantly on modeling prices continuously over time, leading to the concept of continuous-time models.

Brownian motion emerges as a natural counterpart to the discrete random walk in continuous time. Originating as a physical model describing the motion of tiny particles suspended in a liquid, Brownian motion has been subject to study in the physics domain since the early 20th century, with luminaries like Albert Einstein contributing to its theoretical framework (1905). In the work of Black Scholes, and Merton (1973), geometric Brownian motion serves as the foundational mathematical model for price dynamics. They astutely recognized the profound connection between Brownian motion and a sophisticated mathematical theory known as stochastic or Itoˆ calculus, named after the Japanese mathematician Kiyosi Itoˆ, who pioneered its development in the 1940s.

Early studies investigating market price behaviour suggested the presence of long-term dependencies in price correlations, implying the existence of stochastic memory in asset returns. Such a phenomenon challenges the efficiency of markets in swiftly arbitraging new information, potentially allowing abnormal profits based on past information. This perspective contrasts with the efficient market hypothesis. However, subsequent studies employing advanced statistical methodologies have found that financial returns in major liquid markets exhibit no significant memory, as evidenced by correlation functions of price increments approaching zero for time periods longer than 15 minutes (Cheung and Lai, 1993; Crato, 1994; Fung and Lo, 1993; Irwin, Zulauf, and Jackson, 1996).

MATERIALS AND METHODS

In the introduction, we establish the concept of *Zt*, wherein *Bt* represents a standard Brownian motion and *As* denotes a continuous or piecewise continuous process. If ∫0*tAs*2*ds* is finite for every *t*, then *Zt* emerges as a square integrable martingale. However, if this condition is not satisfied, the stochastic integral may fail to exhibit martingale properties.

Theorem 1: A continuous process M_t adapted to the filtration $\{F_t\}$ is called a local martingale on [0, T) if there exists an increasing sequence of stopping times $\tau_1 \leq \tau_2 \leq \tau_3 \leq \ldots$ such that with probability one lim *j→∞ τj = T and for each j, Mtj = Mt∧Tj is a martingale.*

For continuous martingales, the optional sampling theorem comes into play, asserting that under specific conditions, it's impossible to gain an advantage in a fair game. The subsequent theorem encompasses two versions particularly valuable for practical applications.

Theorem 2: Suppose Z_t is a continuous martingale and T is a stopping time, both with respect to the filtration $\{F_t\}$. If $M_t = Z_{t\wedge T}$, then M_t is a continuous martingale with respect to $\{F_t\}$. In particular, $E[Z_{t\wedge T}] = E[Z_0]$

Theorem 3: Suppose there exists $C < \infty$ such that for all t, $E[Z_{t}^{2}{}_{\wedge T}] \leq C$. Then if $P\{T < \infty\} = 1, E[Z_T] = E[Z_0]$. Suppose Z_t is a continuous martingale and there exists $C < \infty$ such that $E[|Z_t|] \leq C$ for all t. Then there exists a random variable Z_∞ such that with probability one $\lim_{t\to\infty} Z_t = Z_\infty$

*Theorem 4: Suppose Z_t is a continuous square integrable martingale, and let Nt = max*0≤*s*≤*t*[*Zt*]

Then for every a > 0 $p\{N_t \geq a\} \leq E[\frac{Z_t^2}{a^2}]$

Feynman-Kac Formula

suppose the stock price evolves according to a geometric Brownian motion.

$dX_t = mX_tdt + \sigma X_tdB_t)$ (9)

International Journal of Mathematics and Statistics Studies, 12 (3), 1-16, 2024

Print ISSN: 2053-2229 (Print),

Online ISSN: 2053-2210 (Online)

Website: https://www.eajournals.org/

Publication of the European Centre for Research Training and Development -UK

Let's consider a scenario where at a future time T, we have the opportunity to purchase a share of the stock at a price S. We'll only exercise this option if the stock price at T is greater than or equal to S. Consequently, the value of the option at time T is represented by $F(XT)$, where $F(X)=(X-S)+=\max(X-S,0)$.

We introduce an inflation rate r, indicating that x dollars at time t in the future are equivalent to $e-rtx$ in current dollars. Denoting $\phi(t,x)$ as the expected value of this option at time t, measured in dollars at time t, given that the current stock price is x.

$$\varphi(t,x) = E[e^{-r(T-t)}f(X_t)|X_t = x] \tag{10}$$

The Feynman-Kac formula provides a partial differential equation (PDE) for this quantity. Generalizing further, we assume Xt satisfies the stochastic differential equation (SDE) $dXt=m(t,Xt)dt+\sigma(t,Xt)dBt$, with initial condition $X0=x0$, and there exists a payoff $F(XT)$ at some future time T. Additionally, we introduce an inflation rate $r(t,x)$, where if Rt denotes the value at time t of $R0$ dollars at time 0, we have $dRt=r(t,Xt)Rtdt$ and $Rt=R0\exp\{\int 0tr(s,Xs)ds\}$.

If $\phi(t,x)$ represents the expected value of the payoff in time t dollars given $Xt=x$, then

$$\phi(t,x) = E[exp\{-\int_t^T r(S,X_s)ds\}F(X_t)|X_t = x] \tag{11}$$

Theorem 5: (Feynmann-Kac formula). Suppose X_t *satisfies* $dX_t = m(t,X_t)dt + \sigma(t,X_t)dB_t, X_0 = x_0$ *and* $r(t,X_t) \geq 0$ *is a discounting rate. Suppose a payoff F at time T is given with* $E[|f(X_T)|] < \infty$. *If* $\varphi(t,x)$, $o \leq t \leq T$ *is defined as in (3.3.3), and* $\varphi(t,x)$ *is c0 in t and* c^2 *in X then* $\varphi(t,x)$ *satisfies the PDE*

$$\partial_t\phi(t,x) = -m(t,x)\partial_x\phi(t,x) - \frac{1}{2}\sigma(t,x)^2\partial_{xx}\phi(t,x) + r(t,x)\phi(t,x), \tag{12}$$

for $0 \leq t < T$, *with terminal condition* $\varphi(T,x) = F(x)$

In Theorem (5), we rely on the assumption that ϕ exhibits adequate differentiability. Conditions on the coefficients and the payoff function F can be provided to ensure this requirement.

Continuous Martingales

Previously, we noted that Brownian motion stands as the sole continuous martingale.

Theorem 6: *Suppose that* M_t *is a continuous martingale with respect to a filtration* $\{F_t\}$ *with* $M_0 = 0$, *and suppose that the quadratic variation of* M_t *is the same as that of standard Brownian motion,*

$$< M >_t = \lim_{n\to\infty} \sum_{j<2^n t} [M(\frac{j+1}{2^n}) - M(\frac{j}{2^n})]^2 = t \quad (13)$$

Then every $\lambda \in <$,

$E[exp\{i\lambda M_t\}] = e^{-\lambda^2 \frac{t}{2}}$ sketch of proof. Fix λ and let $f(x) = e^{i\lambda x}$.

Note the derivatives of f are uniformly bounded in x. Following the proof of $It\hat{}0^0s$ formula we can show that

$$f(M_t) - f(M_0) = N_t + \frac{1}{2}\int_0^t f''(M_s)ds = N_t - \frac{\lambda^2}{2}\int_0^t f(M_s)ds \quad (14)$$

where N_t is a martingale. In particular, if $r < t$,

$$E[f(M_t) - f(M_r)] = \frac{1}{2}E[\int_r^t f''(M_s)ds] = \frac{-\lambda^2}{2}\int_r^t E[f(M_s)]ds. \quad (15)$$

If we let $G(t) = E[f(M_t)]$, we get the equation

Black-Scholes Equation

Consider the Black-Scholes equation for option pricing, focusing on a single stock with a price $S(t)$ varying over time. Almagren (2002) suggests that the option's market value derived from this stock should be a function of S and t, denoted as $V(t,S(t))=d(t)$. In finance, an asset's profitability is chiefly characterized by its rate of return. To model the fluctuations in the return on a stock, we employ a geometric Brownian motion. A process exhibits geometric (or exponential) Brownian motion if its logarithm follows a Brownian motion, indicating that random variations occur solely as fractional changes. This process is expressed by the differential equation $dS=aS(t)dt+bS(t)dw(t)$, where a and b are constants, and $w(t)$ is a Brownian motion. We leverage Ito's lemma to analyze this process

$$dD = [V_t + aSV_s + \frac{b^2S^2}{2}V_{ss}]dt + bSV_s dw(t) = [V_t + \frac{b^2S^2}{2}V_{ss}]dt + V_s ds \quad (16)$$

Let's consider an investor who possesses a combination of the stock and its corresponding option within their portfolio.

$P(t) = N_1(t)S(t)+N_2(t)D(t)$.

The differential is

$$dP = N_1 ds + N_2 dD = N_1 ds + N - 2[V_t + \frac{b^2 S^2}{2} V_{ss}]dt + N_2 V_s ds] \tag{17}$$

Malliaris (1987) then presents the insightful idea of maintaining a ratio between stocks and derivatives, where

$$\frac{N_1}{N_2} = -V_s (this is known as a delta hedge), so that N_1 ds + N_2 V_s ds = 0$$

Then we have:

$$dP = N - 2[V_t + \frac{b^2 S^2}{2} V_{ss}]dt \tag{18}$$

This asset is entirely unrelated to Brownian motion, as it lacks any dw(t) term, whether explicitly stated or implied. Consequently, it can be regarded as devoid of risk. According to the efficient market hypothesis, the return on this risk-free asset should mirror that of any other risk-free asset, such as a government bond. Let's denote the return on the government bond as r(t).Then we have

$$\frac{dP}{P} = \frac{N - 2[V_t + \frac{b^2 S^2}{2} V_{ss}]dt}{N_1 S + N_2} = rdt \tag{19}$$

If we rearrange and normalize so that $N_1 = 1$, thus making $N_1 = V_s$, we get

$$[V_t + \frac{b^2 S^2}{2} V_{ss}]dt = (-V_s S + V)rdt \tag{20}$$

or

$$V_t(t,s) + \frac{b^2 S^2}{2} V_{ss}(t,s) + SV_s(t,s) - rV(t,s) = 0 \tag{21}$$

This equation represents the Black-Scholes model, which is fundamental for pricing options. Up to this juncture, we've delved into the requisite mathematical foundation crucial for comprehending the practical utilization of stochastic calculus within finance—the Black-Scholes option pricing model.

ANALYSIS AND DISCUSSION

Black-Scholes option pricing model

Under a set of significant assumptions, the Black-Scholes option pricing model can determine the non-arbitrage rational price of an option.

Set-up of the Model

If the price per unit of the underlying stock option follows a stochastic process represented by S = {St : t ≥ 0}. Additionally, let's assume that S adheres to the geometric Brownian motion stochastic differential equation (SDE): dSt = St(μdt + σdwt). The distinct solution to this geometric Brownian motion SDE is as follows:

$$S_t = f(t, W_t) = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t},$$

Assuming normalization with S0 = 1, market participants are assumed capable of buying any quantity, including a negative quantity (equivalent to short selling) of the stock without transaction costs. Furthermore, they can purchase any quantity of a risk-free asset, or bond, which maintains a constant continuously compounded interest rate of r. Denoting the price of one unit of the bond at time t by Bt, B satisfies the integral equation dBt = rBtdt, leading to Bt = ert after normalization with B0 = 1.

Negatively buying a bond corresponds to borrowing funds at an interest rate of r. We define a portfolio (at, Bt) as a pair of stochastic processes, both adapted to the filtration of σ-fields generated by Wt, where at and bt represent the number of units of stock and bond in the portfolio, respectively, at time t. The portfolio's value is determined by:

$V_t(a,b) = a_t s_t + b_t B_t.$

The portfolio is self-financing if alterations in its value solely result from fluctuations in the stock and bond prices, without any external inflows or outflows of funds. The self-financing condition is expressed as:

$dV_t(a,b) = a_t ds_t + b_t dB_t$

To price an option, we rely on the no-arbitrage assumption, asserting the absence of risk-free profit opportunities. Our objective is to identify a self-financing portfolio that mimics the payout of the specific option under consideration—a portfolio tailored to the option type. Consequently, the option's value at time t equates to the value Vt(a,b) of the self-financing portfolio (a,b) replicating the option.

European Call option

In this section, we determine the no-arbitrage rational price for a European call option utilizing the Black-Scholes model. The European call option grants the holder the right to purchase one unit of stock at a fixed strike price, K, at a predetermined maturity time, T. The holder can only exercise this option at time T and retains the discretion not to purchase the share.

Consequently, the option's value at time T is represented by max{ST − K,0} = (ST − K,0)+. Since the European call option is exercisable solely at time T, a portfolio (a,b) with value function Vt(a,b) replicates the option if VT (a,b) = (ST − K,0)+ almost certainly. Assuming the existence of a self-financing portfolio (a,b) replicating the European call option, we aim to determine the value function of (a,b) by hypothesizing the presence of a smooth deterministic function u: $<^2 \to <$

$V_t = a_tS_t + b_tS_t = u(T - t, S_t)$ for $t \in [0,T]$

Assumption of the model

Stock price is approximated by geometric Brownian motion

$$S_t = e^{(\mu - \frac{1}{2}\sigma^2)t} + \sigma w_t$$

This assumption holds critical importance for several reasons. Firstly, it encompasses the weaker assumption that *St* follows an Itoˆ process, enabling us to leverage the Itoˆ integral and Itoˆ lemma to solve the stochastic differential equations derived from the model. Secondly, specifying *St* as a geometric Brownian motion entails assuming a constant mean rate of return μ and volatility σ from time t=0 to t=T, thereby defining the nature of the randomness embedded in the stock price. This assumption essentially suggests that the Wiener process provides a suitable approximation of the stochastic behavior observed in stock prices over time.

However, it's important to acknowledge that while modeling the stock price process as a geometric Brownian motion is theoretically convenient, it does not reflect reality accurately. In practice, stock prices cannot take irrational values, thus necessitating that *St* belongs to the set of positive real numbers. This discrepancy with reality poses a challenge, as stock prices are bounded and finite, contrary to the assumption of unboundedness in geometric Brownian motion.

Furthermore, the proof that the Black-Scholes portfolio replicates the European call option relies on the assumption that for continuous random variables $P(X=x)=0$ for all x in the real numbers. However, in reality, stock prices must be elements of the rational numbers, a countable subset of the real numbers. Consequently, the assertion that $P(ST=K)=0$ is no longer straightforwardly true when considering rational strike prices.

Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)
Website: https://www.eajournals.org/
Publication of the European Centre for Research Training and Development -UK

Deviation of a Discrete-Time Portfolio from the Self-Financing Property: Correction

Suppose we break the interval [0,T] into $n \in N$ intervals each of length $\frac{T}{n}$ and that financial market participants perform transactions only at time t $\in \{\frac{T}{n}\}$ for j ∈ {0,1,...,n}. at this times t, financial market participants solve the Black-Scholes portfolio equations:

$a_t = \varphi(g(T - t, S_t))$, and $b_t = -Ke^{-rt}\varphi()h(T - t, S_)$, and adjusting their holdings of stock and bond accordingly. Because the stock price S_t changes between $t_j = \frac{jT}{n}$ and $t_{j+1} = \frac{(j+1)T}{n}$, in the absence of instantaneous re-balancing by financial market participants, the value of the portfolio they must acquire at time tj+1,Stj+1, atj+1+βtj+1btj+1, may differ from the value of the current portfolio they hold before executing rebalancing transactions at time tj, Stj+1atj+βtj+1btj. We can visualize this rebalancing transaction as the participant selling the portfolio held at time tj, (atj,btj), at the prices at time tj+1, and subsequently purchasing the desired portfolio at time tj+1, (atj+1 ,btj+1), at those same prices. The disparity between the proceeds gained from selling (atj,btj) and the funds needed to purchase (atj+1,btj+1) constitutes the correction over the time interval (tj,tj+1], shaping our measure of the deviation of the portfolio from the self-financing property.

The correction Cj of a portfolio over the time interval (tj,tj+1] is expressed as:

Cj=Stj+1(atj+1−atj)+βtj+1(btj+1−btj).

A positive Cj indicates that the participant must inject additional funds from an external source into the portfolio to execute the required transaction at time tj+1, while a negative Cj indicates that the participant must withdraw funds from the portfolio to execute the required transaction.

The correction Ctk of a portfolio up to time tk is defined as follows:

$\sum_{j=0}^{k-1} C_j$.

It's important to highlight that Ctn represents the correction of a portfolio up to the option's maturity time, denoted as T. We symbolize Ctn as C. The subsequent sections delve into the examination of corrections for various Black-Scholes portfolios in depth. In each model, time units, such as intervals of length I, are employed to denote a day.

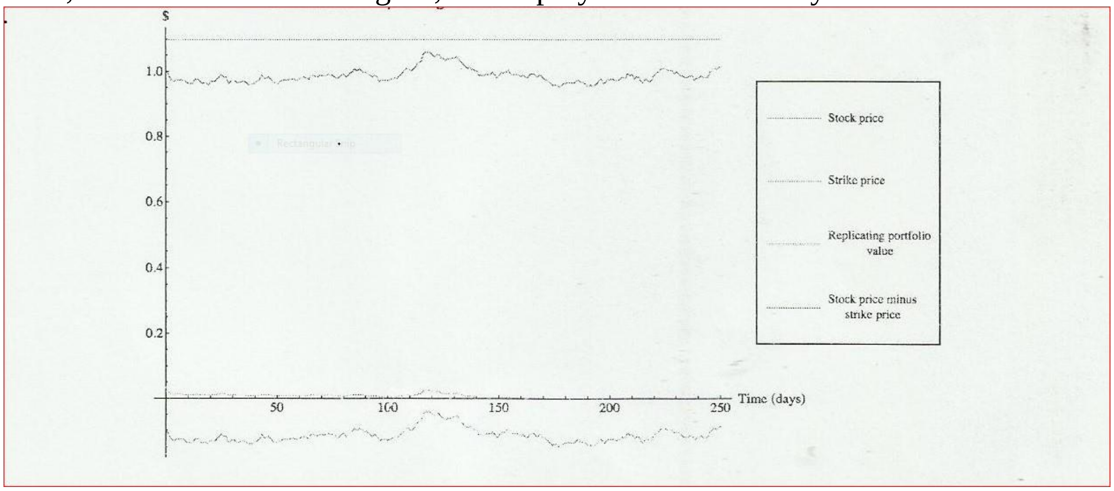

Figure 1: Stock price and value of replicating portfolio for given sample path W_0 of S_t

Source: Greg W. (2012)

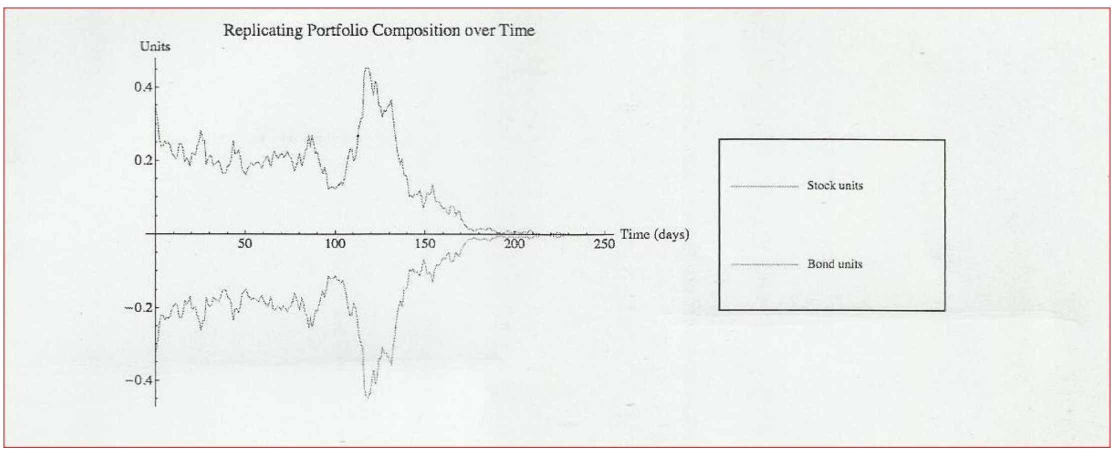

Figure 2: Composition of replicating portfolio over time for sample path W_0 of S_t

Source: Greg W. (2012)

Correction of sample paths of S_t

The first model we built to study the correction of Black-Scholes portfolios simulates one sample path of the stock price process S_t, and tracks the correction of the portfolio over time. The model allows for the specification of K,T,μ, r,and n. It then calculate S_{tj},a_{tj},b_{tj}, for $j \in \{0,1,...,n\}$ and C_{tj}, for $j \in \{1,2,...n\}$.

The following is a sample output from the model with $S_0 = 1, B_0 = 1, K = 1.1 T = 250, n = 250$, and annualized $\mu = 0.1, \sigma = 0.05$ The sample path of S_t that generate this output is the stock price charted in figure 1.

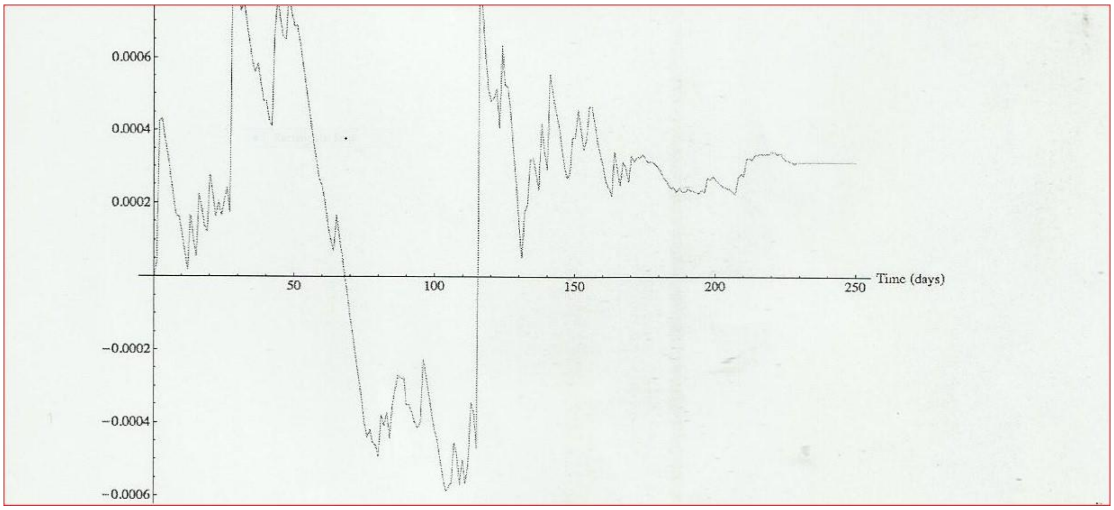

Figure 3: correction over time for same sample path W_0 of S_t Source: Greg W. (2012)

Figure 1, 2, and 3 depict the composition, value, and correction of the replicating Black-Scholes portfolio, respectively. Notably, in this particular sample path of St, ST, where < K, aT = bT = 0, the correction C is observed to be positive. This indicates that to replicate the European call option, financial market participants would have needed to inject additional funds into the portfolio from an external source.

From the analysis presented, we can draw an observation concerning the correction and the construction of the replicating portfolio in discrete time steps: notable increases in stock price within a given time interval necessitate significant adjustments in the number of stock units held in the portfolio. Consequently, such instances are more likely to incur a positive correction over the interval. This phenomenon's sensitivity is influenced by factors such as the remaining time to maturity, the relative stock price compared to the strike price, and σ.

This model not only enables us to scrutinize the correction across specific sample paths of St but also serves as a foundational framework for conducting numerical analyses of the correction across multiple sample paths which has solution.

$$G'(t) = \frac{-\lambda^2}{2} G(t), \quad 54G(0) = 1$$

$$G(t) = e^{-\lambda^2 \frac{t}{2}}$$

Consequently, Ys embodies a standard Brownian motion. To delve deeper into understanding the behavior of the correction across multiple sample paths, we've devised a second model. This model aims to elucidate how the correction, denoted as C up to time T, behaves across various sample paths for differing values of n, the ratio measuring the frequency of portfolio rebalancing per day.

Intuitively, we anticipate that as the frequency of rebalancing increases, C should converge toward 0 more frequently. To bridge the discrete-time model with the continuous-time Black-Scholes model, particularly for large n, we aim to demonstrate that C tends toward 0 as n approaches infinity. Our model allows for the customization of parameters such as K, T, μ, σ, r, the sequence of n time steps under consideration, and the number of iterations performed for each n. Following 50 iterations for each n ∈ {30, 600, 600, 900, 1200, 1500, 1800, 2100, 2400, 2700, 3000}, using the specified parameters: So = 1, β0 = 1, K = 1.02, T = 30, and annualized μ = 0.07, σ = 0.1, r = 0.05, our findings indicate a discernible trend where C tends to approach 0 more closely as n increases. However, the rate of convergence diminishes for larger values of n, and it's crucial to note that this simulation lacks the rigor of a formal proof.

Furthermore, our aspiration to analyze the correction's behavior for exceedingly large values of n encounters challenges, as computing limits necessitates a balance between the number of iterations feasible at each n and the range of n values amenable to analysis

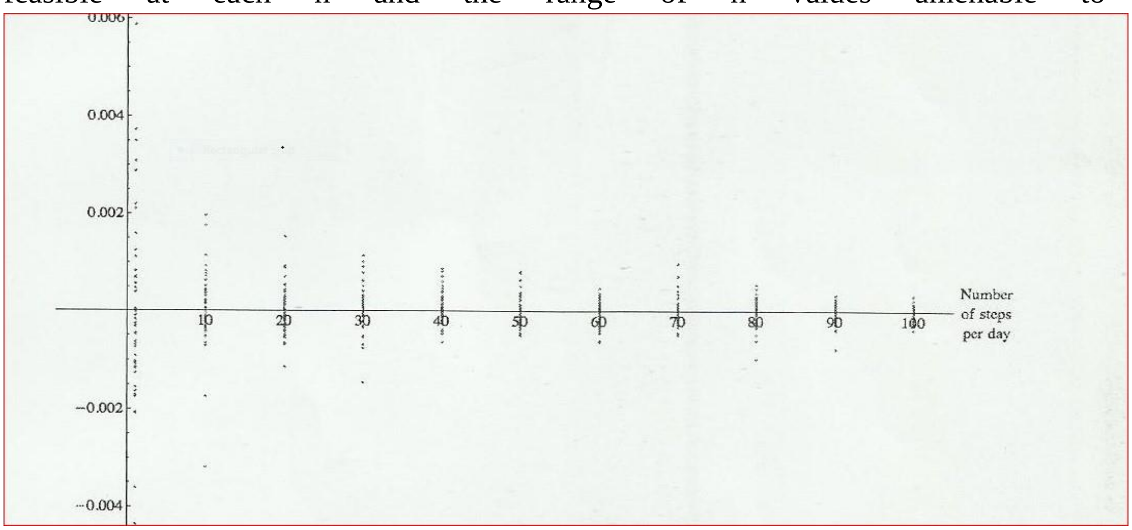

Figure 5.1: correction over several sample paths and values of $\frac{T}{n}$

Source: Greg W. (2012)

Correction of portfolios based on historical stock price data

To gain insights into how corrections manifest in the real-world implementation of a Black-Scholes replicating portfolio, we've developed a third model. This model constructs a replicating portfolio and meticulously tracks its correction using a specified series of historical stock price data. Our methodology involves utilizing daily closing prices of stocks and setting the parameter n equal to T, enabling financial market participants to adjust their portfolios once per day. We simulate the creation of the Black-Scholes portfolio over a one-year period (T = 250), mirroring the approximate number of trading days in a year. The parameters employed for all examples remain consistent throughout this process.

$S_0 = 1, \beta_0 = 1, T = 250, K = 1.1$, and annualized r = 0.05.

Creating a Black-Scholes portfolio without relying on geometric Brownian motion, but rather on historical price data, presents a challenge. This is because the Black-Scholes portfolio's effectiveness hinges on the parameter σ, representing volatility within the price process. To apply the Black-Scholes formula at any given time t, we must make assumptions regarding σ for times s > t. We explore two distinct methods for determining an assumed volatility, both rooted in the historical volatility of the stock being analyzed.

The first method, dubbed the constant volatility approach, entails calculating the historical volatility of the stock over the preceding six-month period (125 days) leading up to t=0. We then assume that σ maintains this constant value throughout the time span t ∈ [0,T].

In contrast, the second method, termed the rolling volatility approach, involves computing the rolling historical volatility of the stock for the six months (125 days) preceding each time point t. This rolling volatility value is then utilized in determining the Black-Scholes portfolio at time t.

Historical volatility calculations (represented as v) adhere to established financial literature conventions and follow a standardized methodology described below.

Given daily stock prices $s_t, s_{t+1}, \ldots s_{t+n}$ let $r_i = \frac{log(s_{t+i})}{log(s_{t+i-1})}$ for i = 1, 2, .., n. Let

$r = \frac{1}{n}\sum_{i=1}^{n} ri.$ Then the daily volatility v_{daily} is given by

$$v_{daily} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_i - r)^2} \quad \sqrt{}$$

and we annualised the volatility by multiplying by $\sqrt{250}$, $annual = \sigma = \sqrt{250}\, v_{daily}$.

CONCLUSION

We study the correction of different Black-Scholes portfolios using time units like intervals of length I. We build a model that simulates one sample path of the stock price process and tracks the correction of the portfolio over time. The model calculates the composition, value, and correction of the replicating Black-Scholes portfolio. The results show that large increases in stock price over a time interval require large increases in the number of stock units held in the portfolio and are more likely to necessitate a positive correction over the interval. This phenomenon is sensitive to the time remaining to maturity, the stock price relative to the strike price, and σ. The study focuses on numerical analysis of correction over several sample paths using a model to calculate the correction up to time T,C. A second model is created to better understand the behaviour of C for different values of n. The model allows for the specification K, T, μ,σ,r, the sequence of n, and the number of iterations for each number n of times steps. The output shows a clear trend to be more likely to be closed to 0 as increases. However, the rate at which C becomes more likely to be close to 0 is lower for large n. The study also aims to study the behavior of the correction for very large values of n, but computing limits necessitate a trade-off between the number of iterations and values of n. The third model creates a replicating portfolio and documents the correction of that portfolio for a given series of historical stock price data.

Acknowledgements
The following people are greatly appreciated for their contributions to the success of the research, Tertiary Education Trust Fund (TETFund); Professor Adeoluwa, V.O., The Vice Chancellor of Bamidele Olumilua University of Education, Science and Technology, Ikere Ekiti and Centre for Research and Development (CERAD), Bamidele Olumilua University of Education, Science and Technology, Ikere Ekiti.

REFERENCES

Almgren R.(2002). Financial derivatives and partial differential equations. The American mathematical monthly, 109(1); 1-12

Andrzej K. (2017) Hedging the Black-Scholes call option, from the wolfram Demonstration Project. http://demonstrations. Wolfram.com/Hedging the Blackscholes call option/.

Black F and Scholes M. (1973). The pricing of option and corporate liabilities. The Journal of political Economy,81(3; 637-654).

Figlewski S. (1994) Forecasting Volatility Using Historical Data, New York University Stern school of Business Finance Department, Working paper series, http://w4.stern.nyu.edu/finance/docs/wp/1994/pdf/wpa94032.pdf

Gikhman I. and Skorokhod (1969) Introduction to the Theory of Random Processes. W.B.Saunders company.

Gregory F. L.(2014)*Stochastic Calculus: An Introduction with Applications* American Mathematical society.

Grimmett G.and Stirzaker D. (2001) Probability and Random Processes, Third Edition, Oxford University press

It$\hat{o}$ Kiyosi (1951) *On Stochastic Differential Equations.*. Memoirs, American Mathematical Society (4); 1-51

Malliaris A (1983) Calculus in Financial Decision Making SIAM Review, 25(4); 481- 496

Mikosch T. (June 2004) Elementary Stochastic Calculus with Finance in View, Advanced Series on Statistical Science & Applied Probability, Volume 6, World Scientific Publishing Co.Pte.Ltd.

Oksendal B (2003) Stochastic Differential Equations : An Introduction with Applications. Springer-verlag.

Rudin W.(1976) Principle of Mathematical Analysis. Mcgraw-Hill inc

Sahoo S. K. and Mishra M. N (2012) Journal of Mathematics(IOSRJM) 2 38-42 www.iosrjournals.org

Shreeve S. E. (2004). Stochastic Calculus for Finance. Springer-verlag

Steele J. M. (2001) Stochastic calculus And Financial Applications. Springer-verlag

Tong Y. L. (1990) The Multivariate Normal Distribution, Springer Series in Statistics, Spriger-verlag, New York.

White G.(2012) Stochastic Calculus and Applications to Mathematical Finance.

Williams college, Williams town, Massachusetts