International Journal of Mathematics and Statistics Studies
Vol.10, No.2, pp.1-25, 2022
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

LOGICAL EQUIVALENCE FAILURE

James H. Hughes*
*Writer, Albuquerque, New Mexico, USA

ABSTRACT: *As a point of mathematical logic, this paper proves that the contrapositive is not logically equivalent to the conditional, which disproves the "proof by contradiction" that has been used in mathematics for over a hundred years. The paper then provides suggestions for why the "proof by contradiction" may have been derived in error.*

KEY WORDS: Conditional, contrapositive, logic, mathematics, proof by contradiction, Venn diagram.

PROPOSITION: THE CONTRAPOSITIVE IS NOT LOGICALLY EQUIVALENT TO THE CONDITIONAL.

When it is difficult to prove a proposition directly, mathematics often uses the proof by contradiction, or the indirect proof, as another method (Proof by Contradiction 2003). However, a few mathematicians, called "constructionists," have argued the method is invalid. They hold that mathematical proofs should follow the pattern set by Euclid, where proofs are constructed from axioms.

The steps for a proof by contradiction are:

First, assume the opposite of your conclusion. For example, to prove "the primes are infinite in number," assume that the set of prime numbers is finite in number, with a finite number of elements of size n.

Second, use this assumption to derive new consequences until you find one that is the opposite of your premise, or contradicts it. In the above example, you would seek to show there exists a prime number not counted in the initial set of n prime numbers, making it a counterexample to your assumption.

Third, conclude your assumption must be false, so that the opposite assumption is true, and hence your original proposition, called the conditional, is true.

This method makes sense by noting you are creating a direct proof of the contrapositive of your original proposition of, if A (condition), then B (conclusion). In other words, you are proving that if not B (conclusion), then not A (condition), a statement that is called the

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

contrapositive. Since the contrapositive is always logically equivalent to the conditional, your original proposition is considered to be proven or true.

The contradiction forces us to reject our assumption of not B because all our other steps are justified. The only "mistake" we made was in our assumption of not B. An indirect proof establishes that since the opposite conclusion is not consistent with the premise, then the original conclusion must be true.

However, in the above example, the counterexample may contain an inductive process able to extend the set of prime numbers by one element, which may be used to show they are infinite in size using a direct proof.

References may not always clearly distinguish between the proof by contradiction and counterexample. The proof by contradiction uses two negations. It assumes the opposite of the desired conclusion (not B) to find a contradiction (not A) as a consequence, which it equates to proving the original proposition. In contrast, the counterexample uses a single negative example to disprove a proposition.

It may be helpful to illustrate the relationship between the conditional and contrapositive, and the converse and inverse, two other logical forms based on the conditional, by using "If it is raining, then I carry my umbrella" as an example of a conditional statement (Conditional Statements 2003).

Logical Form	Symbols	Statement
Conditional	$A \rightarrow B$	If it is raining, then I carry my umbrella.
Converse	$B \rightarrow A$	If I carry my umbrella, then it is raining.
Inverse	$\sim A \rightarrow \sim B$	If it is not raining, then I do not carry my umbrella.
Contrapositive	$\sim B \rightarrow \sim A$	If I do not carry my umbrella, then it is not raining.

The proof by contradiction relies on the logical equivalence of the contrapositive of "If I do not carry my umbrella, then it is not raining" to the conditional of "If it is raining, then I carry my umbrella." This equivalence does not extend to the converse or inverse, which are not considered logically equivalent.

The lack of logical equivalence between the conditional and the converse or inverse is seen in the counterexample of how I sometimes carry my umbrella when it is not raining to protect myself from the hot sun. Since I sometimes carry my umbrella when it is not raining, this disproves the converse. And since when it is not raining I sometimes carry my umbrella, this disproves the inverse.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

To understand why the converse and inverse are not logically equivalent to the conditional, it may be helpful to more closely examine the logic behind them.

Converse: $B \rightarrow A$ If I carry my umbrella, then it is raining.

The converse reverses the order of the cause and effect. In other words, where the conditional starts with the cause or condition of it raining, which results in my carrying my umbrella, the converse reverse the condition and its result to say, "If I carry my umbrella, then it is raining."

While it may sound plausible, carrying my umbrella does not cause it to be raining. The converse sounds plausible because it implicitly assumes that I have checked the weather and seen it is raining in making my decision to carry my umbrella. However, deciding to carry my umbrella based on the condition that it is raining assumes the conclusion that the converse seeks to prove. This is not acceptable in mathematics. Proofs are not supposed to assume their conclusion.

Mathematical proofs derive their conclusion by using a process of reasoning, which frequently involves algebra or other symbolic manipulations, based on a set of axioms or assumptions that are accepted by time and experience.

Returning to the converse, my decision to carry my umbrella does not cause the condition of it raining. But this is what the converse claims in reversing the order of cause and effect, or condition and result.

The converse claims that carrying my umbrella causes it to be raining without checking the condition of the weather since checking the weather would assume the conclusion of it raining. Commonsense tells us that this is incorrect. This is why the converse is not logically equivalent to the conditional. Cause and effect flow from cause to effect, and are not freely interchangeable.

In other words, when B is a result of condition A, the converse or reverse conclusion that condition A is the result of B does not hold as a matter of general principle. The idea that effect can become cause and cause can become effect as a matter of logical equivalence, which is the logic of the converse, is not accepted.

Symbolically, $A \rightarrow B$ is not equivalent to $B \rightarrow A$. In other words, the logic operator is not commutative.

Inverse: $\sim A \rightarrow \sim B$ If it is not raining, then I do not carry my umbrella.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

The inverse applies the ~ or "not" to both sides of the conditional like an algebraic constant to arrive at $\sim A \rightarrow \sim B$. Although the inverse sounds plausible, as noted in the counterexample, they are times I carry my umbrella when it is not raining.

The inverse assumes that the opposite condition or not A (~ A) produces the opposite result or not B (~ B) as a general principle. While the condition of not A can be expected to produce a different result than B, it may not necessarily be the opposite of B. Its cause and effect relationship needs to be reworked. This is why the inverse is not logically equivalent to the conditional.

The opposite condition requires the cause and effect relationship to be reworked in order to determine its result rather than assuming that the opposite condition necessarily results in the opposite effect.

Symbolically, the logic operator does not distribute the ~ or "not" like an algebraic constant. In other words, the logic operator is not distributive.

Interestingly, the contrapositive represents both the inverse applied to the converse, and the converse applied to the inverse. In other words, the combination of the inverse and converse applied to each other produces $\sim B \rightarrow \sim A$.

For example, applying the inverse to the converse starts with the statement "If I carry my umbrella, then it is raining." It applies the ~ or "not" to both sides. Applying the "not" to "If I carry my umbrella" and "then it is raining" results in "If I do not carry my umbrella, then it is not raining" or the contrapositive $\sim B \rightarrow \sim A$.

But since the inverse is known not to be logically equivalent to the conditional, this statement, or the contrapositive, is not logically equivalent to the conditional of "If I carry my umbrella, then it is raining." In other words, the negative condition of the statement "If I do not carry my umbrella" does not necessarily imply the negative conclusion of "then it is not raining."

Moreover, since "If I carry my umbrella, then it is raining" is known not to be logically equivalent to the original conditional of "If it is raining, then I carry my umbrella" or that the fact that I carry my umbrella does not determine the condition of it raining, neither is it reasonable to suppose the fact that I do not carry my umbrella cause the condition of it not raining.

The converse reverses the order of cause and effect. The inverse assumes the opposite condition produces an opposite effect. If one is wrong, the other does not correct it as a matter of general principle, so their combination does not correct each other.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Since both the converse and inverse are not logically equivalent to the conditional, it is not obvious how their combination results in a statement that is logically equivalent to the conditional. Reversing the natural flow of cause and effect, and using the opposite result or conclusion as a cause or condition that is assumed to produce the opposite cause or condition does not naturally correct each other. So the contrapositive is not logically equivalent to the conditional.

Alternatively, applying the converse to the inverse starts with the statement "If it is not raining, then I do not carry my umbrella" and reverses the condition and result. This gives the statement "If I do not carry my umbrella, then it is not raining" or the contrapositive $\sim B \rightarrow \sim A$.

But since the converse is known not to be logically equivalent to the conditional, this statement, or the contrapositive, is not logically equivalent to the conditional of "If it is not raining, then I do not carry my umbrella." In other words, reversing the cause and effect so "I do not carry my umbrella" becomes the condition does not necessarily produce the result that "it is not raining."

It can also be asked how changing the condition of "If I carry my umbrella" to "If I do not carry my umbrella" causes the weather to not be raining when it is generally known that carrying my umbrella does not cause the weather to be raining. These negatives argue that the contrapositive is not logically equivalent to the conditional.

In other words, reversing the order of cause and effect and distributing the "not" like an algebraic constant to the conditional results in a statement, called the contrapositive, that is not logically equivalent.

Now let's examine the contrapositive under the counterexample of how I sometimes carry my umbrella when it is not raining.

Contrapositive: $\sim B \rightarrow \sim A$ If I do not carry my umbrella, then it is not raining.

In the counterexample, I sometimes carry my umbrella when it is not raining. Thus, when I carry my umbrella it may or may not be raining, and when it is not raining I may or may not be carrying my umbrella, so when I do not carry my umbrella does not determine that it is not raining.

Since the condition of it not raining occurs independently of whether I carry or do not carry my umbrella, it cannot be said that my not carrying my umbrella determines that it is not raining. This violates the contrapositive, which says, "If I do not carry my umbrella, then it is not raining."

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

This is an argument of juxtaposition. It shows that the condition of it not raining occurs independently of when I carry my umbrella. When I do not carry my umbrella does not cause the weather to not be raining. This shows that the contrapositive is not logically equivalent to the conditional.

The contrapositive appears to work because it uses a sophism. Like the converse, it implicitly assumes that I check the weather before I decide not to carry my umbrella. But without checking to see what the weather is, the rain is unpredictable based on whether or not I carry my umbrella. My decision to carry or not carry my umbrella does not determine the weather, but is used in response to the weather.

In other words, the rain is a condition, which prompts me to carry my umbrella. But I sometimes carry my umbrella when it is not raining. Whether or not I carry my umbrella does not determine the condition that it is not raining. The fact that I carry my umbrella is only an indicator that it may be raining. Thus, it cannot be concluded that if I do not carry my umbrella, then it is not raining.

Where in the conditional, the condition of rain causes me to carry my umbrella in a true cause and effect relationship, the contrapositive substitutes my not carrying my umbrella for the condition of it not raining to show it is not raining when umbrellas do not cause the weather to be raining or not raining.

This is a mistake. The condition of it raining or not raining does not depend upon any action taken by the subject with his umbrella. In the conditional, the subject recognizes a condition, which causes him to act. The subject and his action do not necessarily affect the condition, and generally have no effect on it.

SECOND DEMONSTRATION

A second demonstration is offered, which tests the logical equivalence of the contrapositive to the conditional, and the conditional to the contrapositive based on their conditions.

To demonstrate the logical equivalence of the contrapositive to the conditional, the logic operator would ask under what conditions I carry my umbrella to show the conditional of "If it is raining, then I carry my umbrella" and test them to see if they are equivalent. The conditional allows two conditions of it raining or not raining, but speaks only to the condition of it raining.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

CASE ONE

Under the condition that it is raining, if I carry my umbrella, I satisfy the conditional. But this requires rewording the condition and conclusion to say, "If it is raining, then I carry my umbrella." This rewording can be done since it is given that it is raining, which determines that I carry my umbrella.

Under the condition that it is raining, if I carry my umbrella, I can appear to satisfy the contrapositive since the presence of these conditions does not contradict it since it speaks to the condition of when I do not carry my umbrella. These two conditions appear under the form of its contrapositive, the contrapositive of the contrapositive, or the conditional. But the contrapositive does not produce these conditions as a logical conclusion as it is inappropriate to assume the conclusion in proving a proposition.

Thus, while the appearance of the two conditions does not contradict the contrapositive, neither does the contrapositive produce them as a logical conclusion. Since the absence of a logical construction does not provide one, the contrapositive cannot be shown to be logically equivalent to the conditional.

CASE TWO

Under the condition that it is raining, if I do not carry my umbrella, I violate the conditional since the conditional says that if it is raining, then I carry my umbrella, and it was given that it is raining.

Under the condition that it is raining, if I do not carry my umbrella, I violate the contrapositive, which argues that my not carrying my umbrella causes it to not be raining. While the contrapositive does not speak to the condition of it raining, the rain violates its conclusion using its condition that I do not carry my umbrella. Thus, under case two, since it can be argued that both the conditional and contrapositive are violated, the contrapositive appears to be logically equivalent to the conditional.

While two more cases can be developed under the condition that it is not raining, they are omitted since the conditional only speaks to the condition of it raining.
But if the cases are developed, they show that the conditional and contrapositive allow the conditions but do not produce them as a logical conclusion. Since the absence of a logical construction does not provide one, it cannot be shown that they are logically equivalent.

In conclusion, since in one of two cases it cannot be shown that the contrapositive is logically equivalent to the conditional, it can be concluded that the contrapositive is not logically equivalent to the conditional.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

STEP TWO

To demonstrate the logical equivalence of the conditional to the contrapositive, the logic operator would ask under what conditions it is not raining to show the contrapositive of "If I do not carry my umbrella, then it is not raining" and test them to see if they are equivalent. The contrapositive allows two conditions, of carrying my umbrella or not carrying my umbrella, but speaks only to when I do not carry my umbrella.

CASE ONE

Under the condition that I do not carry my umbrella, if it is not raining, I can appear to satisfy the conditional since it does not speak to the conditions of when it is not raining and I do not carry my umbrella. The presence of these conditions does not contradict it. On the other hand, neither does the conditional produce these conditions as a logical conclusion.

Under the condition that I do not carry my umbrella, if it is not raining, I satisfy the contrapositive. This requires rewording the condition and conclusion to say that if I do not carry my umbrella, then it is not raining. The statement can be reworded since it is given that I do not carry my umbrella, and it is not raining.

Thus, while under case one the conditional allows the conditions of the contrapositive to appear, it does not produce them as a logical conclusion, so it cannot be shown to be logically equivalent to the contrapositive.

CASE TWO

Under the condition that I do not carry my umbrella, if it is raining, I appear to violate the conditional, which says "If it is raining, then I carry my umbrella." While the conditional does not speak to the condition of when I do not carry my umbrella, it does speak to the condition of it raining to require me to carry my umbrella, which violates the given conditions.

Under the condition that I do not carry my umbrella, if it is raining, I violate the contrapositive, which says that if I do not carry my umbrella, then it is not raining, and it was given that I do not carry my umbrella.

Thus, under case two, since it can be argued that both the conditional and contrapositive are violated, the conditional appears to be logically equivalent to the contrapositive.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

While two more cases can be developed under the condition that I carry my umbrella, they are omitted since the contrapositive only speaks to the condition of when I do not carry my umbrella. But if the cases are developed, they will show the contrapositive and conditional allow the conditions but do not produce them as a logical conclusion. Since the absence of a logical construction does not provide one, it cannot be shown that they are logically equivalent.

In conclusion, since in one of two cases it cannot be shown that the conditional is logically equivalent to the contrapositive, it can be concluded that the conditional is not logically equivalent to the contrapositive.

ELEMENTS OF MATHEMATICS

To gain some insight on the idea that the contrapositive is logically equivalent to the conditional, it may be helpful to discuss some of the basic building blocks or elements of mathematics, and three main laws of logic upon which this idea is based, which include the use of the truth function.

One of the first elements of mathematics is the axiom. Axioms are elementary statements, so obvious they are taken to be true, used to build mathematical systems of thought or the branches of mathematics.

One of the first branches of mathematics is Euclidean geometry, the geometry of flat surfaces. Using a compass and straightedge, it deals with the construction of geometrical figures, and the relationship of angles, especially in a triangle. Other geometries deal with curved surfaces.

Another early branch of mathematics is arithmetic. It deals with the addition, subtraction, multiplication, and division of numbers. Closely related to arithmetic is algebra. It deals with the rules of arithmetic, order of operations, exponents, and use of symbols, which represent unknown variables or constants, to solve equations.

An offshoot of algebra is analytic geometry, which uses the Cartesian system of (x, y) coordinates to describe the horizontal and vertical displacement of a point in a plane with respect to an origin of (0, 0). Linear algebra, another offshoot, solves systems of linear equations. The branch of calculus takes the limit of the slope and summation functions. Probability and statistics deal with the analysis of data and patterns.

As a rule, axioms are few in number, independent, and strong. Axioms are few in number since mathematical systems tend to be more powerful if they are based on a few axioms, which represent the distillation of a mathematical or physical system to its key elements.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Axioms are independent in the sense they are not able to be derived from each other, while being necessary to describe a system. They are strong in the sense of being able to serve as building blocks in the construction of proofs and theorems.

Axioms tend to be chosen that accurately represent the physical universe so their resulting proofs and theorems are useful in solving practical problems. But axioms are sometimes chosen that are speculative or ideal in order to explore thought problems.

A second element of mathematics is logic. It deals with the construction of proofs and the analysis of arguments. It proves a proposition through the skillful use of axioms and existing proofs and theorems in a line of reasoning, which is based on a series of steps that follow each other in an orderly manner.

Similar to how two points define a line, a line of reasoning uses a series of steps that follow each other to take a reader from a setting of basic ideas to a conclusion, which is stated by the proposition of a proof or theorem.

As a rule, logic seeks consistency. This drive for consistency sometimes results in rules that may seem arbitrary such as the rule of algebra that prohibits division by zero or any algebraic expression such as (x-x) that is equal to zero. Otherwise, division by zero lets any number or algebraic expression become equal each to other when they are clearly different.

While the rule that prohibits division by zero is not arbitrary since the operation of division requires a number with value, zero has a prominent role in the number system as the identity element of addition and subtraction, and balance point between the positive and negative numbers.

In other words, mathematics can be more complex than it first appears. The rule that prohibits division by zero shows the complexity of running two different operations of arithmetic, namely addition and subtraction, and multiplication and division, over the same set of numbers, when arithmetic operations affect their system of numbers.

Another element of mathematics, perhaps the most common, is the number. Indeed, an entire branch of mathematics, number theory, is devoted to their study. Numbers are first introduced with the counting or the natural numbers, which include the prime numbers that play a key role in number theory. The addition of zero gives the whole numbers. Adding the negatives of the natural numbers gives the integers.

Rational numbers are formed by fractions of natural numbers. Irrational numbers are formed by algebraic operations such as the square root of two, and include transcendental

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

numbers such as π and e. Imaginary numbers use the square root of negative one as a multiplier, and are found in electrical engineering and physics.

Another element of mathematics is the skill used to solve word problems, which are practical problems involving word and sentence descriptions. This skill is developed by acquiring basic skills in arithmetic and algebra, and learning to read and interpret word problems, often by drawing a picture or diagram to help describe the problem.

Word problem skills provide a model of logical construction with their three main steps of first analyzing a problem and its set up, the writing of equations to describe a problem in precise algebraic terms, and calculation of a solution. Intermediate results are usually displayed for checking.

Word problem skills are often applied to science and engineering, and other fields, even to analyze issues of public policy such as the problems of a space program. These types of advanced problems usually involve the analysis of arguments, and explanation of errors in logic.

TRUTH FUNCTION

Widely used in propositional logic, the truth function evaluates whether a statement is true or false. Returning a value of true or false, its range consists of two elements, letting it be represented by two numbers such as 0 and 1, or a number system of base two. Number systems of base two are often encountered in computers and electrical engineering, where they may denote whether a switch is turned off or on.
The truth function is based on the idea that a statement can be evaluated as true or false. To maintain consistency, it requires that an impartial observer use its system of truth, and applies its frame of reference on a consistent basis.

However, the truth function could consider evaluating whether a statement is unknown or incapable of being definitely determined, as some statements seem to involve a degree of uncertainty. To include a category for unknown, the truth function would need a range with three elements, which could be represented by three numbers such -1, 0, and 1.

When applied to compound statements, which are easily divisible into components, each able to be evaluated as true or false, the range of the truth function returns a string of true or false based on its evaluation of each component, so that its return of information is more complex than a simple true or false.

When applied to mixed statements, which consist of parts both true and false, but not directly divisible into components, just as water can become a solution that is mixed with

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

a material or other liquid, the truth function's range or return of information is not able to distinguish between the state of being mixed, or being both true and false at the same time, but in a different manner or proportion.

Evaluating mixed statements would be like filtering polluted water, which may require osmosis or distillation. To evaluate these types of statements, the truth function's range, or return of information, may need an explanation for what parts are true or false and in what manner or proportion.

As a practical matter, statements of fact can usually be quickly ascertained as true or false. Opinions and explanations may be more difficult to categorize. Other statements may be compound, or mixed, describe contradictory behavior, or difficult to categorize. In other words, the universe of statements appears to have a range of ease, or difficulty, in being evaluated as true or false.

As a practical matter, science and other fields rely heavily on mathematical functions of the form $y = f(x)$ to explain cause and effect relationships more than the truth function, although they use the idea of the truth function in testing hypotheses. Since science and other fields rely heavily on mathematical functions of the form $y = f(x)$ in explaining cause and effect relationships, mathematical logic could consider changing the truth function to examine the quality of theories and explanations.

Closely related to the truth function is the idea that a statement cannot contradict itself. In mathematical terms, this would be similar to the reflexive law of algebra, but applied to a statement. The idea of non-contradiction is important since most mathematical systems are based on the idea of internal consistency.

However, statements sometimes do contradict themselves as a matter of style, or the artful manipulation of opinion. In other cases, the contradiction is unintentional or a sign of a weak argument. Regardless, a statement that contradicts itself should not necessarily be dismissed since it may provide useful information, be used to challenge an audience, or reveal a speaker's motivations or standards.

LAW OF NON-CONTRADICTION

In the field of logic, three laws appear with some prominence, which may be used to develop the idea that the contrapositive is logically equivalent to the conditional. Since it was shown that the contrapositive is not logically equivalent to the conditional, it may be helpful to review these laws of logic, to see whether they are indeed obvious and suitable as laws of logic, or a sound foundation to build the argument that the contrapositive is logically equivalent to the conditional.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

The first law of logic, the Law of Non-Contradiction, says that a statement cannot contradict itself. Stated in terms of using the truth function, it says that a statement cannot be both true and false at the same time.

However, a statement can be both true and false at the same time. For example, a statement may be divisible into components some of which are true while others are false. Or a statement may have components mixed with varying degrees of being true or false. Or a statement may describe contradictory behavior, or be difficult to categorize as true or false as its veracity may be ambivalent, in dispute, unknown, or incapable of being definitely determined.

Examples of statements both true and false at the same time often appear in politics and economics. Even Chinese fortune cookies, whose sayings typically convey good wishes for the future, are often worded in such general terms that they may be considered to be both true and false at the same time.

Another class of contradictory statements appears in sports and weather forecasts, and probability and statistics. Indeed, by definition, probabilities give a precise value for the expected occurrence of a future event as an explicit formulation of being both true and false at the same time.

While after an event, forecasts and probabilities may be evaluated as true or false, at least in a retrospective sense, they retain the element of truth, in a prospective sense, by having accurately measured the expectation of a future event.

Another class of statements, both true and false at the same time, may be found in the conditional or "if, then" statement that a given condition leads to a specific result such as the conditional "If it is raining, then I carry my umbrella." Even if a conditional is true, there may be exceptions to its rule such as when it is raining I do not carry my umbrella since it is missing and instead wear a raincoat so that the truth of a conditional, which gives a general rule, depends upon its circumstances.

In addition, a conditional operates under its condition. Unless its condition is met, it does not represent a fact, and may be said to be false. So, while a conditional may be true as a rule, its truth depends upon its actual circumstances and operation, making it both true and false at the same time.

Statements both true and false at the same time appear in physics. For example, where the speed of light is a universal speed limit, scientists have subdivided light to where part of it travels faster than the speed of light while another part does not travel as fast. While this contradiction about the speed of light being a universal speed limit may be resolved by

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

adding the faster piece to the slower piece in order to retain a complete picture of the light, it represents a change in the former rule.

Other contradictions appear in describing light as a wave or particle. While these contradictions are generally explained by observing how the behavior of light seems to change from a wave to a particle at the atomic level, light possesses a dual nature, both wave and particle, making it contradictory in behavior.

In other words, evaluating even factual statements as true or false may require skill in determining what parts are true or false, or mixed, ambivalent, in dispute, or unknown. And some statements may be incapable of being definitely determined just as electron shells are sometimes used to approximate the location of an electron.

In addition, in science and other fields, new knowledge and understanding can change what is viewed as true and false over time.

In summary, by using the truth function, the Law of Non-Contradiction does not seem to satisfy a practical viewpoint that recognizes how some statements have components that are both true and false at the same time, mixed, describe contradictory behavior, or are difficult to categorize, being ambivalent, in dispute, unknown, or incapable of being definitely determined.

DE MORGAN'S LAWS

After the Law of Non-Contradiction, a second law of logic appears in De Morgan's laws, a pair of rules that distribute the negation or "not" or ~ sign in propositional logic and in set theory, which have practical uses in electrical engineering and computers much like Boolean algebra (De Morgan's laws).

In propositional logic, De Morgan's laws distribute a negation over a disjunction and conjunction. His first law distributes the negation of a disjunction into a conjunction with two negations. His second law distributes the negation of a conjunction into a disjunction with two negations.

In set theory, De Morgan's laws distribute the negation or "not," often called the complement of a set, over the intersection and union of two sets. The complement of a set consists of all the elements that lie outside the set. It is not limited to the elements that are found outside the set, which may represent an incomplete set if an incomplete search function or incomplete universe, inadequately defined, is used.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

De Morgan's first law distributes the complement of the intersection of two sets into a union of their complements. His second law distributes the complement of the union of two sets into an intersection of their complements.

Using the notation of set theory, where ∩ denotes the intersection of two sets, called p and q, U denotes their union, and ~ denotes the complement, De Morgan's laws may be written as:

1. $\sim (p \cap q) = \sim p \cup \sim q$
2. $\sim (p \cup q) = \sim p \cap \sim q$

De Morgan's first law states that the complement of the intersection of sets p and q is equal to the union of the complement of set p with the complement of set q. His second law states that the complement of the union of sets p and q is equal to the intersection of the complement of set p with the complement of set q.

In set theory, De Morgan's laws may be pictured by drawing a diagram, called a Venn diagram, which uses a box to represent a universe of elements, and two circles in the box to represent two different sets. The area the circles intersect represents the intersection of the two sets with their common elements. The area the circles enclose represents the union of the two sets, which combines their elements into a larger set.

Venn diagram: (diagram with the universe, two circles, intersection, and union)

De Morgan's first law may be illustrated by drawing a Venn diagram with sets p and q, taking the complement of their intersection, and comparing it to a Venn diagram of the union of the complement of set p with the complement of set q.

Two Venn diagrams: (diagrams) the complement of the intersection of sets p and q, and the union of the complement of set p with the complement of set q. By inspection, the two diagrams show an equal area.

De Morgan's second law may be illustrated by drawing a Venn diagram with sets p and q, taking the complement of their union, and comparing it to a Venn diagram of the intersection of the complement of set p with the complement of set q.

Two Venn diagrams: (diagrams) the complement of the union of sets p and q, and the intersection of the complement of set p with the complement of set q. By inspection, the two diagrams show an equal area.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Using De Morgan's laws, the Law of Non-Contradiction may be used to derive the third law of logic, the Law of the Excluded Middle, which says that a statement must be either true or false. In other words, the Law of the Excluded Middle says there is no middle ground on whether a statement is true or false.

However, the Law of the Excluded Middle does not necessarily follow from the Law of Non-Contradiction since the Law of Non-Contradiction allows a larger universe to appear:

Universe of the Law of Non-Contradiction

Statements that are true
Statements that are false
Statements that are not true, but false
Statements that are not true, but not false
Statements that are not false, but true
Statements that are not false, but not true

The Law of Non-Contradiction also allows statements that are difficult to categorize for being ambivalent, in dispute, unknown, or incapable of being definitely determined as it is not apparent these statements are both true and false at the same time. However, it excludes statements that are both true and false at the same time, arguing they do not appear as a result of logic.

To shorten the list, statements that are not true but not false, and not false but not true may be placed into the category of not true or false. Statements that are not true but false, and not false but true may be placed into the categories of false and true, respectively. With these changes, the Law of Non-Contradiction allows the following universe to appear:

Universe of the Law of Non-Contradiction

Statements that are true
Statements that are false
Statements that are not true or false
Statements that are difficult to categorize for being ambivalent, in dispute, unknown, or incapable of being definitely determined.

In contrast, the Law of the Excluded Middle allows the following universe to appear:
Universe of the Law of the Excluded Middle

Statements that are true
Statements that are false

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Compared to the Law of Non-Contradiction, the Law of the Excluded Middle omits the categories of not true or false, or difficult to categorize for being ambivalent, in dispute, unknown, or incapable of being definitely determined. It represents a smaller universe, filtered into statements that are either true or false.

In contrast, the Law of Non-Contradiction represents a principle of non-exclusion. It does not exclude statements that are not true or false, or difficult to categorize so that it allows a middle ground to appear, which is otherwise excluded by the Law of the Excluded Middle.

VENN DIAGRAM ILLUSTRATIONS

This point about the Law of Non-Contradiction representing a larger universe than the Law of the Excluded Middle may be illustrated by a series of Venn diagrams. The first depicts the typical Venn diagram where two circles, which represent different sets, intersect each other.

Typical Venn diagram: (diagram)

The typical Venn diagram shows that sets tend to have intersections, and unions that are non-trivial, meaning that one set is not a subset of the other. It also shows that, regardless of their exact juxtaposition, sets have boundaries that are well defined, and tend to encompass only part of a universe.

As a further point, the typical Venn diagram shows that a universe of elements, defined by a common trait or characteristic, may be subdivided into at least two sets. Since a set is defined by its elements, or is distinguishable from another set by having at least one element that is not in common, the typical Venn diagram requires a minimum of four elements in order to display a non-trivial intersection between two different sets, and universe with at least one element outside the union of the sets.

In other words, the typical Venn diagram partitions a universe into four distinct areas. These areas consist of the two sets, their intersection, and the universe outside their union. Four elements are needed to distinguish each area, apart from being an area or a set without any elements.

As a further point, a partition or set needs only one point or element to be non-trivial, a slightly different requirement than how a line requires two points for its geometrical construction, as set theory deals with categories and classification.

The typical Venn diagram also implies that elements are able to be distinguished or segregated from each other, so they may be placed into sets.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

SECOND VENN DIAGRAM (subsection heading)

The tendency of sets to have intersections may be illustrated by a second Venn diagram that shows the sets of black and white in a universe of color. Although the sets are stark opposites, they share an intersection in the shades of grey, as grey is a mixture of black and white, with sometimes another color added such as blue.

Second Venn diagram: (diagram)

The second diagram shows how sets tend to encompass only part of their universe, or have a union that tends not to encompass the entire universe, just as the universe of color has many colors, in addition to black and white.

THIRD VENN DIAGRAM (subsection heading)

The first and second Venn diagrams suggest that a third Venn diagram may be drawn the same way, using the set of statements that are true and the set of statements that are false, which shows an intersection between the two sets, and a union that encompasses only part of the universe of statements.
In other words, the third diagram shows an analogy. Just as black and white are opposites, and true and false opposites, their Venn diagrams should be similar, showing the same characteristics of an intersection between the two sets, and a union that encompasses only part of the universe.

In other words, the third diagram is like the first two, but uses the set of statements that are true and the set of statements that are false, which measure the veracity of a statement as a common trait or characteristic.

Like the first two diagrams, the third diagram shows that there is an intersection between the set of statements that are true and the set of statements that are false, just as the sets of black and white share an intersection in shades of grey. In other words, it indicates that some statements may be both true and false at the same time.

The third diagram also shows that the set of statements that are true and the set of statements that are false do not encompass the entire universe, just as not every element in the universe of color has a measurable content of black or white, or some other color. In other words, it indicates that not every statement is either true or false, or has a measurable content of veracity.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Vol.10, No.2, pp.1-25, 2022
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

Third Venn diagram: (diagram)

While the third Venn diagram only suggests that there is an intersection between the set of true statements and the set of false statements, it may be observed that some statements are divisible into components each true or false so they are not necessarily true or false in total, while other statements are mixed, or forecasts or probabilities, or difficult to categorize, which corroborates its suggestion.

In effect, the third Venn diagram is the same as the first two, but depicts the set of statements that are true and the set of statements that are false. While it does not satisfy the Law of Non-Contradiction since it shows an intersection between the two sets, the intersection is readily explained.

In other words, the third diagram represents the principle of non-exclusion found in the Law of Non-Contradiction, since it is inclusive of statements that are both true and false at the same time, statements that are not true or false, and statements that are difficult to categorize for being ambivalent, in dispute, unknown, or incapable of being definitely determined. In a sense, it represents the Law of Non-Contradiction, but without the strictures of the truth function.

FOURTH VENN DIAGRAM (subsection heading)

The fourth Venn diagram illustrates the Law of Non-Contradiction. To do this, it adjusts the third diagram to show no intersection between the set of statements that are true and the set of statements that are false. To emphasize this point, it shows the circles as separate instead of sharing a common point or boundary.

Fourth Venn diagram: (diagram)

FIFTH VENN DIAGRAM (subsection heading)

The fifth Venn diagram depicts the Law of the Excluded Middle. Its universe is split between the set of statements that are true and set of statements that are false. While like the fourth diagram it does not show an intersection between the two sets, it shows a common boundary between the sets, which splits the universe into two pieces.

In other words, the fifth diagram shows the Law of the Excluded Middle as the set of statements that are true and the set of statements that are false from the fourth diagram, transposed into a universe that consists of just the two sets.

Fifth Venn diagram: (diagram)

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

With its common boundary, the fifth diagram suggests that there is a clear demarcation between the set of statements that are true and the set of statements that are false, without any middle ground, the point of the Law of the Excluded Middle.

However, the fifth diagram appears to misrepresent the sets of true and false statements since the first three Venn diagrams suggest that those sets have an intersection and do not encompass the entire universe of statements. In other words, just as the sets of black and white in the universe of color have an intersection and do not encompass the universe, so should the sets of true and false in the universe of statements have an intersection and not encompass the universe.

Since the series of Venn diagrams show a progressively smaller universe, especially between the third and fifth Venn diagrams, the Law of the Excluded Middle represents a smaller universe than the Law of Non-Contradiction.

In particular, the Law of the Excluded Middle excludes statements that are both true and false at the same time. It excludes statements that are not true or false. And it excludes statements that are difficult to categorize for being ambivalent, in dispute, unknown, or incapable of being definitely determined. The Law of the Excluded Middle excludes these statements as a matter of definition rather than having demonstrated their non-existence within a universe of statements.

DE MORGAN'S LAWS, CONTINUED

Returning to the Law of the Excluded Middle, and how it is said to follow from the Law of Non-Contradiction, which says "A statement cannot be both true and false at the same time," De Morgan's first law may apparently be used to apply the "not" in "cannot" to take the negation or complement of the intersection of the set of statements that are true with the set of statements that are false.

This application of De Morgan's first law results in the following formula where T stands for the set of statements that are true, and F stands for the set of statements that are false:
$\sim (T \cap F) = \sim T \cup \sim F$

The right side of the equation, or $\sim T \cup \sim F$, equals the union of the complement of the set of statements that are true with the complement of the set of statements that are false. Applying the complement directly to the interior of a set, or to its elements, the right side equals the union of the set of statements that are not true with the set of statements that are not false. This assumes the complement passes through the boundary of the set without any interference or friction.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Then, by saying the negation of a statement that is not true is false and the negation of a statement that is not false is true, the right side equals the union of the set of statements that are false with the set of statements that are true.

Reversing the order of the sets, which is permitted under the operations of union and intersection, the right side equals the union of the set of statements that are true with the set of statements that are false. In other words, the right side of the equation, which equals ~ T U ~ F, also equals T U F.

Since the union of the set of statements that are not true with the set of statements that are not false represents all the elements of the Venn diagram, and equals the union of the set of statements that are true with the set of statements that are false, a statement must be either true or false, just as argued by the Law of the Excluded Middle.

However, using the terminology of set theory, the Law of Non-Contradiction may be reworded to say "A statement cannot be both an element of the set of true statements and an element of the set of false statements at the same time." In turn, this may be reworded to say "A statement that is both an element of the set of statements that are true and the set of statements that are false, at the same time, cannot be."

This reworded version of the Law of Non-Contradiction clarifies that the "not" applies to the number of elements found in the intersection between the set of statements that are true with the set of statements that are false. Since such a statement cannot exist, the number of elements found in the intersection is zero, or the null set.

Since moving the "cannot" to the end of the Law of Non-Contradiction gives a clear statement of the law without requiring the use of De Morgan's laws, the application of De Morgan's first law appears to be extraneous. Moreover, the application of his first law to take ~ (T ∩ F) does not properly apply the "not" in "cannot" as it takes the complement of the intersection of the two sets when the Law of Non-Contradiction only enumerates the intersection between the two sets.

In other words, there is a difference between enumerating the intersection of two sets compared to taking the complement of their intersection. Thus, the derivation of the Law of the Excluded Middle from the Law of Non-Contradiction by using De Morgan's laws appears to be in doubt.

Moreover, using the negation of set theory to say a statement that is not true is false, and a statement that is not false is true imposes the unstated assumption that an element of the complement is necessarily the opposite in trait or character when the complement of a set,

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

which is defined as the elements outside a set, does not require its elements to all be opposite in trait or character.

In other words, the search function of the negation of a statement that is not true is not limited to statements that are false, but includes statements that are both true and false at the same time, or not true or false, or difficult to categorize.

So, the complement of the set of statements that are true is not limited to statements that are false, but includes statements that are both true and false at the same time, or not true or false, or difficult to categorize. Likewise, the complement of the set of statements that are false is not limited to statements that are true, but includes statements that are both true and false at the same time, or not true or false, or difficult to categorize.

So, saying a statement is not true does not necessarily mean that it is false, and saying a statement is not false does not necessarily mean that it is true. A statement that is not true or false is an element of the set of statements that are not true or false, without requiring it to be either true or false.

Thus, saying a statement that is not true is false and a statement that is not false is true begs the question of middle ground as it excludes such "middle ground" statements as a matter of definition, rather than mathematical proof.

This "middle ground" includes statements that are not true or false, both true and false at the same time, or describe contradictory behavior, or difficult to categorize for being ambivalent, in dispute, unknown, or incapable of being definitely determined.

THE DOUBLE NEGATIVE

With respect to the use of De Morgan's laws in logic, it may be helpful to take a closer look at the complement of a set, the "not" or negation of set theory, by drawing the Venn diagram for not the intersection of sets p and q by taking the complement of their intersection, which gives an area with one negation.

Venn diagram showing one negation:

Comparing this to the Venn diagram for the union of not p with not q, gives an equivalent area but with three types of negations. The first negation is not p. The second negation is not q. The third negation is a double negation for the intersection of not p and not q.

Venn diagram showing three negations:

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

While the three negations seem to neatly fold into a single negation so that De Morgan's laws may be said to hold, something more needs to be said about the use of the negation in logic.

For example, a conditional statement does not distribute the negation or "not" as seen in the lack of logical equivalence between a conditional and its inverse.

Moreover, where a double negative or "not not" cancel each other out algebraically, so that $\sim \sim p = p$, from a practical viewpoint of English composition, which is an important consideration in mathematical logic, a double negative often conveys a twist in meaning and is generally a weaker statement.

In other words, in logic, $\sim \sim p$ is less than or equal to p.

While a double negative is used at times because of the preceding use of a negative, or because it adds literary flavor, clarity in thought almost always prefers p to the double negative of $\sim \sim p$.

Even when a double negative is indeed mathematically equivalent, as in the rule of algebra that says the product of two negative numbers is equal to a positive number, or that $-1 \times -1 = 1$, from a practical viewpoint, countless errors of algebra creep in while working problems with the product of negative numbers.

It is human nature to become confused with complex communications such as a double negative, both in communicating the intended meaning, and understanding the meaning intended. This carries over into logic, and working of math problems that involve the product of negative numbers, and other types of negations. So that, in logic, unless required, a double negative should generally be avoided.

CONCLUSION

To gain some perspective on the idea that the contrapositive is logically equivalent to the conditional, this paper discussed some of the basic elements of mathematic, and reviewed the three main laws of logic, which start with the Law of Non-Contradiction, and include the use of the truth function.

In essence, the Law of Non-Contradiction says that a statement cannot contradict itself. Stated in terms of using the truth function, it says that a statement cannot be both true and false at the same time. However, in using the truth function the law appears to take on an unstated system of truth to determine whether a statement is true or false, and assumes that an unbiased observer uses the same system of truth.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

While requiring an unbiased observer to use the Law of Non-Contradiction's system of truth appears reasonable, the unstated assumptions surrounding its system of truth are not as obvious as laws of logic.

Then, a review of the Law of Non-Contradiction indicates it is unsuitable for analyzing compound or mixed statements (at least without some adjustment to its range or return of information), probabilities, conditional statements, contradictory behavior, and other types of statements such as the sayings often found in Chinese fortune cookies.

In other words, by using the truth function, the Law of Non-Contradiction appears to take on unstated assumptions that are not obvious as laws of logic, and appears to be unsuited for use in evaluating a variety of statements. As a result, it does not appear to hold as a law of logic.

As a second point, where the Law of the Excluded Middle is generally accepted and said to be derived from the Law of Non-Contradiction by using De Morgan's Laws, set theory suggests otherwise by its listing of set categories, and the common use of Venn diagrams. In other words, set theory does not support the Law of the Excluded Middle's blanket categorization of statements as either true or false.

Moreover, the derivation of the Law of the Excluded Middle seems to misapply De Morgan's first law since there is a difference between enumerating the intersection of two sets compared to taking the complement of their intersection.

As a third point, saying that a statement that is not true is false and a statement that is not false is true, based on the negation of set theory, appears to be erroneous.

The negation or complement of the set of statements that are true and set of statements that are false do not automatically default to the set of statements that are false and the set of statements that are true, respectively, but equals the set of statements that are not true, and the set of statements that are not false.

In other words, a statement that is not true is not necessarily false, and a statement that is not false is not necessarily true. Again, where in number theory the negative of a number is the opposite, in set theory the negation of a set, or its complement, is simply defined as the elements outside the set. While the complement of a set includes any opposites, it is not limited to them.

Finally, as a note of caution, the double negative, while equivalent in a mathematical sense, is inefficient at communication in English composition and prone to induce errors so that care is needed in using it in logical constructions.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Vol.10, No.2, pp.1-25, 2022
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

In summary, while this section only provides perspective on the three main laws of logic and does not examine how the contrapositive is logically equivalent to the conditional by using these laws, as currently stated or applied these laws do not seem to provide a sound foundation to examine the logical equivalence of the conditional to the contrapositive.

In other words, without a sound foundation in laws of logic, mathematics may be expected to show a failure of logical equivalence by claiming that the contrapositive is logically equivalent to the conditional.

In conclusion, since mathematics seeks consistency, logic might better use the principle of non-contradiction to seek consistency rather than use the truth function. This idea of consistency might say a statement should not contradict itself, unless it does so in a different manner, which can be explained.

ACKNOWLEDGEMENTS

The author gratefully acknowledges the kind assistance of the following reviewers. Katherine Hughes and Ashkan Zeinaizadeh (version dated April 28, 2014), Carl Hime (version dated May 4, 2016), and Siobhan May (version dated November 17, 2016). And minor editing since then.

CONFLICT OF INTEREST

I declare I have no conflict of interest.

BIBLOGRAPHY

"Proof by Contradiction," Making Mathematics, Education Development Center, 2003.
"Conditional Statements," Making Mathematics, Education Development Center, 2003.
"The Law of Excluded Middle," An Introduction to Philosophy, Stanford University, downloaded 2014.
"Law of excluded middle," Wikipedia, last modified January 3, 2014.
"De Morgan's laws," Wikipedia, last modified April 6, 2014.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13