International Journal of Mathematics and Statistics Studies
Vol.10, No.1, pp.47-71, 2022
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

THE IDENTITY CRISIS (ELEMENT)

James H. Hughes*
*Writer, Albuquerque, New Mexico, USA

Citation: James H. Hughes (2022) The Identity Crisis (Element), *International Journal of Mathematics and Statistics Studies*, Vol.10, No.1, pp.47-71

ABSTRACT: *To prove that addition and multiplication require different identity elements, this paper introduces the idea of a multiplicative union and the different symmetries and dimensions involved.*

KEY WORDS: Dimension, identity element, multiplicative union, set theory, symmetry, zero.

INTRODUCTION

Duality such as the conversion of matter into energy, and energy into matter, or the behavior of light as a wave and particle, is found throughout nature. Duality is also found in mathematics with its need for two distinct identity elements, zero and one, the identity elements for addition and multiplication. This duality in mathematical identity elements appears in information technology, with its reliance on zero and one, used for programming computers at a basic level.

The idea of a duality in mathematical identity elements was developed from a recent paper (Logical Equivalence Failure 2016), which noted how the drive for consistency in mathematics results in the seemingly arbitrary rule that prohibits division by zero, the identity element for addition.

In other words, the rules of algebra suggest how the different operations of arithmetic require different identity elements, which also serve as building blocks for programming computers at a basic level, and involve the transfer of information, as may be explained using set theory. To help understand the duality in mathematical identity elements, the following proposition is offered.

PROPOSITION: ADDITION AND MULTIPLICATION REQUIRE DIFFERENT IDENTITY ELEMENTS.

Definition:

An identity element is an element of a set that when used in an operation with another element of the set, returns a value of the other element, and has this property for every element of the set, including itself.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

For example, when zero, the identity element for addition, is added to another element from a set of numbers, the result is the other element.

Symbolically, for any element a, a + 0 = a.

Identity elements have at least two properties. One property is that identity elements are unique, meaning that only a single identity element is associated with a given operation.

Theorem: an identity element is unique.

Suppose there are two identity elements, A and B, for a given operation. Using the operation of addition as an example, for any element a:

a + A = a, and a + B = a.

Since both equations give the same result, a + B may be substituted for a in the right hand side of the first equation, which results in the following equation:

a + A = a + B.

Since A and B are both identity elements, this result holds for every element of the set, including when element a is equal to A and B, so that A + A = B + B.

Since the identity element returns a value of itself, A + A = A, and B + B = B.

Substituting A for A + A, and B for B + B, gives A = B.

Since the two identity elements A and B are equal to each other, they are the same, making the identity element unique.

While the same result may be obtained by subtracting element a from both sides of the equation of a + A = a + B, this method of proof avoids the need to introduce the operation of subtraction.

A similar proof may be established using multiplication. In essence, the proof establishes that since both identity elements return the same value of an element, an equation may be written where the result of the operation using identity element A with element a equals the result of the operation using identity element B with element a. Because an identity element returns a value of itself in the operation, A and B may be shown to equal each other, using element a as a point of intersection.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Another property of an identity element is that it follows the commutative property of its operation. If an operation is commutative, the identity element may be used in any order. But if an operation is not commutative, the identity element follows the order that is established by the operation.

For example, since addition is commutative, for an element a, a + 0 = 0 + a = a. However, since subtraction is not commutative, a – 0 = a, but 0 – a = -a, a mirror image of element a, with a negative or reverse orientation.

In other words, under subtraction, the identity element functions as an identity element if it is subtracted from the other element. But when another element is subtracted from the identity element, subtraction returns a mirror image of the other element, which has a different sense of orientation than the original element.

An element that is a mirror image of another element generally has the same value in terms of magnitude, but lies in a different or opposite direction.

Since its result depends on the order of its elements, subtraction is not commutative. Its property of not being commutative illustrates how operations that reverse the result of an operation that is commutative are usually not commutative since their result depends on the order of their elements.

From another point of view, an identity element makes itself invisible. Used in an operation with another element, the operation returns the value of the other element so it is not apparent that the operation has even been performed.

From another point of view, an identity element represents a point of equilibrium or a point of origin. Physical systems tend to have points of equilibrium like the balance on a scale of weights, or a point of origin used to measure distance or time, or some other physical quantity.

For example, the Cartesian system of (x, y) coordinates uses a two dimensional representation of zero or (0, 0) as a point of origin. Polar coordinates also use zero as a point of origin.

However, mathematical systems that consist of an operation that is closed over a set of elements do not always include an identity element. In other words, an identity element may suffer an identity crisis in being excluded from a set just as how the set of natural numbers, which is closed over the operation of addition, excludes zero, the identity element for addition.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

As a result, an identity element may need to be constructed as a new element, which lies outside a set, just as zero, the identity element for addition over the set of natural numbers, needs to be constructed as an element that lies outside the set. Zero may be constructed by using subtraction, the reverse operation of addition, which reverses the flow or direction of its elements.

In other words, since a reverse operation reverses the work of the operation, it is able to restore or return a mathematical system to its point of balance, origin, or equilibrium, and in so doing, construct its identity element.

On the other hand, when an identity element already exists in a set, its identity as an identity element may be established by observation, just as how it may be observed that the set of natural numbers, which is closed over multiplication, includes one as its multiplication identity element.

OPERATIONS

Mathematical systems that consist of sets and operations are created. In particular, an operation does work between two elements of a set, which results in another element of the set. For a mathematical system that initially consists of a single or base element and operation, the system may duplicate the base element to perform an operation, which either results in a new element or the same element.

By duplicating the initial or base element of a set, a mathematical system that consists of a single element and operation may create a set of elements, over which the operation can freely operate. This is seen in how addition may be used to create or generate the set of natural numbers in their natural, consecutive order by using one as a base element, and can operate freely over the set.

By adding one to a duplicate of itself, addition generates two as a new element. Then by repeating the process of adding one to the prior result, addition is able to generate the set of natural numbers in their natural, consecutive order.

In other words, by using an operation in an orderly process, a single or base element can serve as a building block to create a set of elements, over which the operation can freely operate and is closed. An operation is closed over a set when applying the operation to any two elements of the set results in another element of the set.

In contrast to its ability to generate the set of natural numbers using one, addition is unable to generate any elements using zero. When it uses zero, addition does not perform any work that results in a different element.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Since addition is unable to create zero by using any elements from the set of natural numbers, zero is created by the introduction of subtraction as its reverse operation, which is able to generate zero as an identity element and the set of negative integers as a mirror image of the set of natural numbers. Subtraction does this using a similar process to generate the natural numbers, where one is subtracted from itself repeatedly.

In other words, addition and subtraction are not only basic operations of arithmetic, they are able to generate the set of natural numbers and the set of negative integers over which they, and other operations operate. With these two sets in mind, it may help to recall that numbers are discrete elements, which are ordered and uniform in composition, just as how the set of natural numbers are depicted on a number line in order with uniform spacing.

In other words, the number line gives a geometrical presentation of the set of natural numbers in their natural order where each element has the same composition and spacing, built from the same element like building blocks. The elements are all multiples of one, or consist of one as their base element, bounded together.

Operations have properties. One property is whether an operation is commutative, or independent of the order of its elements, just as addition is commutative, where for any two elements, a and b, $a + b = b + a$. The commutative property of addition may be explained by its increasing nature, which combines two elements in a manner that increases their total value, according to the value of each element.

From a geometrical point of view, addition joins together two displacements on a line, whose result is another displacement with a length that is equal to the length of the two displacements. Since addition combines two displacements in a manner that increases the length of their total displacement, its length is independent of the order of the two displacements. Its elements may be freely interchanged without affecting its result. Moreover, the commutative property of addition holds whether its elements have a positive or negative orientation.

In contrast, subtraction is not commutative since it is a reverse operation. Changing the order of its elements switches their flow or orientation, and changes its result, so it is not commutative. Its result depends on the order of its elements.

Another property of an operation is whether it is inductive, meaning it can be applied to multiple elements, where the result of a calculation with two elements is applied to a third element, and so on, until the operation completes its calculation over all the elements, in an orderly manner.

If an operation is commutative, it will also be inductive. For example, since addition is commutative, $a + b = b + a$ = sum (of $a + b$). Induction interjects a third element, c where

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

(a + b) + c = sum + c. Since addition is commutative, sum + c = c + sum, which gives the same result for both sides of the equation.

Operations are ordered. Higher order operations build upon or are explained in terms of a basic operation, just as how multiplication may be explained in terms of addition. Unless otherwise indicated by parentheses, higher order operations such as multiplication are usually performed before basic operations such as addition since they are considered more powerful.

A higher order operation may be viewed as more efficient way of performing a basic operation. For example, multiplication may be viewed as more efficient way of adding the same number repeatedly. But while a higher order operation may be more efficient than a basic operation, it typically operates over a set that has been created by a basic operation.

The idea that operations are ordered and have different roles such as creating elements versus algebraic manipulations, and have different efficiencies as they operate over sets of numeric elements, a key factor in the design of information management systems, is offered as an alternative to the seemingly abrupt introduction of ring theory in abstract algebra, which is based on the two operations of addition and multiplication (Herstein, *Topics in Algebra*).

The view that a basic operation may be used to create the elements of the set that it operates over is offered as an alternative to the introduction of group theory in abstract algebra, which is based on a single operation (Herstein, *Topics in Algebra*).

For example, compared to addition, as multiplication operates over the set of natural numbers, it does not create any new elements. Moreover, it is unable to replicate the set of natural numbers. At best, multiplication may use the prime numbers to replicate part of the set of natural numbers in a process that is complicated, and does not replicate the natural numbers in their natural, consecutive order.

In other words, while multiplication may be more efficient than addition in performing certain types of calculations, it is not as powerful in its ability to generate elements. Based on the idea of addition, multiplication cannot replace it, while addition may replicate its numeric result.

Where multiplication is often explained in terms of addition, division is often introduced as the reverse operation of multiplication, just like how subtraction is often introduced as the reverse operation of addition.

In other words, the operations of arithmetic are paired. Subtraction is the reverse operation of addition, and division is the reverse operation of multiplication. The work that

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

subtraction does reverses the work of addition, and the work that division does reverses the work of multiplication.

As a reverse operation, subtraction is introduced after addition, and its result depends on the order of its elements, making it non-commutative, but it has the same identity element. Likewise, as a reverse operation, division is introduced after multiplication, and its result depends on the order of its elements, making it non-commutative, but it has the same identity element.

OPERATIONS OF ARITHMETIC

The proposition that addition and multiplication require different identity elements is based on the idea that they are different, although related, operations at a fundamental level.

To show this, a uniqueness theorem will be introduced as an intermediate step:

Proposition: addition and multiplication are different operations.

This theorem is motivated by the idea that multiplication is a higher order operation than addition, or a faster, more efficient way to add the same number repeatedly.

To show this, an example will be given in the multiplication of three times five. To mimic the multiplication, three fives may be placed into an array of (5, 5, 5), and added together. The addition is performed sequentially, going from right to left, so a total of two additions are performed: 5 + 5 + 5 = (5+5) + 5 = 10 + 5 = 15.

The array may also be depicted as:

$$\begin{array}{r} 5 \\ 5 \\ \underline{+\ 5} \\ 15 \end{array}$$

If the multiplication were stated as five times three, the array would consist of five threes or (3, 3, 3, 3, 3), and require a total of four additions. So, while the additive array mimics the result of a multiplication, it requires more calculation. In other words, multiplication is more efficient than addition since it gives the same result with only one calculation.

Recognizing the efficiency of higher order operations such as multiplication, algebra gives their calculation priority over basic operations such as addition, unless otherwise indicated by parentheses.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

International Journal of Mathematics and Statistics Studies
Vol.10, No.1, pp.47-71, 2022
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

The difference between addition and multiplication may also be illustrated using the geometrical interpretation of addition as adding two displacements on a line or curve in one dimension, while multiplication uses two displacements in different dimensions to calculate an area.

First illustration: three displacements of five added together result in a displacement of fifteen along a one dimensional line or curve (show both a straight line and curve).

Second illustration: a box with a length of five and height of three when multiplied gives the same numeric result of fifteen but as an area.

By inspection, the two operations give the same numeric result, but with different geometrical meanings. Addition results in a length or displacement in one dimension, while multiplication results in an area in two dimensions.

If a multiplication is performed with three elements, its resulting product may be given the geometrical interpretation of a volume. In other words, multiplication opens the door to calculations in a space with multiple dimensions. In contrast, the result of an addition with multiple elements continues to represent a one dimensional length, or displacement along a line or curve.

Since the geometrical interpretation of addition as a displacement on a line and multiplication as the calculation of an area are both stable, meaning that addition stays in one dimension and multiplication stays in two dimensions, the operations are fundamentally different.

In other words, while addition and multiplication may give the same numeric result, multiplication is inherently more efficient, and may be viewed as occurring over two dimensions while addition occurs over a single dimension.

As an aside, the idea of multiplication representing two dimensions is illustrated by the slide rule, once commonly used in engineering, where several types of number lines are manipulated in a linear manner to mimic multiplication and other algorithms.

SET THEORY

Set theory offers another way to show that addition and multiplication are different operations. The discussion is based on an elementary understanding of set theory with its ideas of organizing elements into sets, and the union and intersection of different sets rather than a specific textbook (Author's education).

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Applying set theory to the example of three times five, multiplication draws the three from a set that defines the number of elements that are placed into an additive array, and draws the five from a set that defines the element that is placed into the array and is duplicated or repeated.

In other words, the additive array is defined by two sets. One set defines the size of the array, or the number of times that a given element is repeated. The other set defines the element that is placed in the array and repeated. The two sets behave in manner similar to how an adjective modifies a noun, and generally precedes the noun it modifies.

The idea the additive array is defined by two sets is like the geometrical interpretation of multiplication as the calculation of an area, which is two dimensional, using an element from a set that represents length along a horizontal axis, and an element from a set that represents height along a vertical axis. In contrast, addition, which combines elements within the same set, has the geometrical interpretation of joining together two displacements on a line or curve, and is one dimensional.

However, when multiplication draws one of its elements from a set of multipliers, or abstract numeric elements, it effectively operates within a single set or dimension, which gives both a numerically equivalent and equivalent set result as addition. In this case, it gives the same set result as addition since it is commutative, both with respect to the order of its elements and sets its elements are drawn from.

In other words, multiplication involves a union between two sets that combines their type of element, units of measurement, or dimension into a new set, which retains the units or dimensions of both sets, and may represents a new unit of measurement.

In other words, numbers have meaning based on the sets that they are drawn from, which defines both their numeric trait or characteristic, and their association with an object, unit of measurement, or dimension.

The operations of arithmetic are transparent with respect to the sets they operate over. For example, numeric comparisons such as equal to, greater than, and less than, occur between elements of the same set just as addition is performed over the same set, or apples are added to apples, and oranges are added to oranges.

In summary, where addition occurs over a single set, multiplication occurs over two sets. While the product of a multiplication folds into a single set if one of its elements is drawn from a set of abstract multipliers, it represents a multiplicative union between two sets, which is fundamentally different than addition, which occurs over a single set.

DIMENSIONAL ANALYSIS

Widely used in science and engineering, dimensional analysis associates numbers with units of measurement or dimensions, which it groups together, following the normal rules of algebra so like numeric terms and units may be canceled in a process that often involves the conversion of units.

Dimensional analysis typically sets up a calculation by placing the units of a number directly after it like a multiplication. Once the units of a calculation are clearly expressed, they may be grouped together, and, if needed, converted into units that are more appropriate or commonly used.

Dimensional analysis is helpful in checking the results of a calculation since mistakes are often made in writing down the correct units and in converting them.

LAWS OF ALGEBRA

The laws of algebra largely restate the properties of addition and multiplication. One prominent property of an operation is whether it is commutative, meaning that the result of its calculation is independent of the order of its elements. For example, addition and multiplication are commutative.

Symbolically: $a + b = b + a$, and $a \times b = b \times a$.

Addition and multiplication are also inductive, meaning they can be applied to more than two elements. Being inductive means that after an operation has completed a calculation or done work between two elements, it may use the result to perform another calculation with another element until it completes its calculation over all the elements, without double counting or omission.

Where it was shown that an operation that is commutative is inductive, mathematical induction may show that an operation that is commutative remains commutative for more than two elements.

For example, since addition is commutative, $a + b = b + a$. Induction interjects a third element, c, where $a + b + c = sum + c$, where $a + b = b + a = sum$. A multiple addition is performed by first calculating $a + b$, and then adding c. Since addition is commutative, $sum + c = c + sum$, so addition is commutative for three elements.

Mathematical induction argues that since the addition was shown to be commutative for three elements as a result of its being commutative for two elements, it is commutative for any discrete number of elements.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

In general, mathematical induction builds a bridge from k elements to k + 1 elements, and argues that since a proposition is true for k = 1 or 2 (the number depending on the number of elements required to give an initial condition), then it is true for an increasing number of elements, to any discrete number.

The property of addition and multiplication being commutative for multiple elements may be verified by their geometrical interpretation. For example, adding two or more displacements on a line gives the same endpoint regardless of the number individual displacements or changes in their order.

Mathematics sometimes states the inductive property of addition and multiplication as the associative law of algebra, where for three elements, a, b, and c:

(a + b) + c = a + (b + c), and (a x b) x c = a x (b x c).

The associative law for addition and multiplication may be derived from their properties of being commutative and inductive.

For example, for addition over three elements, there are six combinations:

a + b + c =
a + c + b =
b + a + c =
b + c + a =
c + a + b =
c + b + a =

Equating a + b + c with b + c + a results in a + b + c = b + c + a.

Since addition is typically performed from right to left, the equation may be rewritten to use parentheses, as: (a + b) + c = (b + c) + a.

Using the commutative property of addition, the right hand side of the equation may be rewritten to substitute a + (b + c) for (b + c) + a, so that (a + b) + c = a + (b + c), which gives the associative law for addition. A similar argument may be used to show that multiplication is associative.

However, mathematics likes to state the associative law independently of an operation being commutative since it shows the use of parentheses in ordering the application of an operation to multiple elements.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Some other common laws of algebra include:

The reflexive law, a = a, which states that an element is equal to itself.

The distributive law of multiplication over addition, where a x (b + c) = (a x b) + (b x c).

The transitive law, where if a = b, and b= c, then a = c, which states a syllogism in mathematical terms.

In contrast, subtraction and division are not commutative since their result is dependent on the order of their elements. Moreover, subtraction is not associative and division does not distribute over addition.

The property of subtraction and division being non-commutative is also inductive, meaning that their result continues to depend on the order of their elements for more than two elements.

However, when subtraction and division operate on more than two elements, there are cases where they give the same result since some pairings of elements may be rewritten to involve addition or multiplication.

For example, for subtraction in three elements, three parings appear:

a – b – c = a – c – b	or	(a – b) – c = (a – c) – b
b – a – c = b – c – a	or	(b – a) – c = (b – c) – a
c – a – b = c – b – a	or	(c – a) – b = (c – b) – a

In other words, subtraction in three elements appears to be commutative when the first element stays the same and the two elements being subtracted change order. This is due to how subtraction distributes over addition. In other words,

a – (b + c) = a – b – c.

Since addition is commutative, a – (b + c) = a – (c + b) = a – c – b.

As a result, a – b – c = a – c – b.

In other words, subtraction in three elements is commutative when the number subtracted is a sum whose elements have the same orientation:

a – (b + c) = a – (c + b)	or	a – b – c = a – c – b
b – (a + c) = b – (c + a)	or	b – a – c = b – c – a

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

c – (a + b) = c – (b + a)   or   c – a – b = c – b – a

But since the three pairings are not equal to each other, the general rule holds that subtraction is not commutative. In other words,

a – (b + c) = a – (c + b) ≠ b – (a + c) = b – (c + a) ≠ c – (a + b) = c – (b + a).

A similar observation may be made about division in three elements, which also gives three pairings where each paring gives a different result, so the general rule holds that division is not commutative.

Other algebras, which use matrices and determinants or complex elements, may follow different rules.

REVERSE OPERATIONS

A reverse operation reverses the work of an operation by reversing the flow of its elements. To do this, it uses for elements the result of the operation and one of the two elements used in the original calculation.

For example, to reverse a + b = c, subtraction reduces the value of c using either a or b:

c – a = b or c – b = a.

Since the result of a reverse operation depends on which of the two original elements it uses to reverse the original calculation, it is not commutative.

As a rule, reverse operations have the same sense of closure as the operation they reverse. But when they operate independently, meaning when the element they reverse no longer depends on reversing an existing calculation, their result may lie outside the set or sets they draw from.

For example, when subtraction operates independently over the set of natural numbers and the number subtracted is larger than the number that it is subtracted from, the result is a negative integer, which lies outside the set of natural numbers.

From another point of view, a reverse operation operates like the operation that it reverses, but using an element with a different or reverse orientation just as how addition may mimic subtraction by adding a negative integer.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Similarly, as division operates independently over the natural numbers, its result may lie outside the set of natural numbers as a fraction, which is not a whole multiple of one, but has a natural number in its numerator and denominator.

From another point of view, since addition and multiplication increase in value or make larger, they are closed over the set of natural numbers since its elements are multiples of the same element, one. But when reverse operations such as subtraction and division, which reduce or make smaller operate independently over the set of natural numbers, they may generate numbers that lie outside the set.

In other words, as subtraction takes away from an element and division splits an element into pieces, they change their number system. Other reverse operations may change their number system just as the square root of two converts a natural number into an irrational number, and the square root of negative one results in an imaginary number.

While reverse operations generally operate with complete freedom over a set, division contains a prominent prohibition against the use of zero as a divisor, since division needs a divisor with value in order to perform work or split apart.

In other words, since zero has no value, division by zero is incapable of splitting apart a number or element so no work is performed and the operation is not defined, a different result than when work is performed using an identity element, whose result is the other element used in a calculation.

While division by zero appears to have a limit since division by increasingly small fractions results in increasingly large numbers, a transition zone may be used in some applications to let a function move past the point of division by zero.

MULTIPLICATION BY ZERO

In contrast to how division by zero is prohibited, multiplication by zero is accepted since zero values often occur in physical systems and bank accounts, and zero is typically used as a point of origin. However, multiplication by zero is an exception to the general rule that multiplication makes larger or increases in value.

Since multiplication by zero always results in a product of zero, regardless of the value of the other element, it is independent of the value or numeric information contained in the other element. Its singular result has the effect of losing or destroying that information, which mimics the clearing of information or resetting often done in computers.

Multiplication by zero always results in a product of zero as a sign that a higher order operation has used the identity element of the basic operation. It must result in a product

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

Vol.10, No.1, pp.47-71, 2022
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

that does not equal the other element. Otherwise, zero would be the identity element for both multiplication and addition, which would mean that the operations are virtually the same, and for any element a, a x 0 = a + 1.

However, in multiplication, zero maintains a sense of consistency as an identity element in that it does not do any work to make another element larger or increase in value, just as it does not do any work to make another element larger or increase in value in addition. With this in mind, zero may be characterized as a neutral element that maintains its trait of being neutral about the value or numeric information of another element in both multiplication and addition.

However, while in addition, zero is neutral in the sense it does nothing to add value to another element, in multiplication, zero is neutral in the sense that the operation is unable to use it in conjunction with the value or numeric information of the other element to perform work.

Multiplication by zero may be explained by using the additive array. First, no matter how many times zero is added to itself, the sum is always zero. Likewise, no matter what element is placed into the array, if the number of elements placed into the array is zero, no work is performed, so the result is a zero sum, or zero.

From a geometrical point of view, multiplication by zero always results in a point with no area since zero is a point that has no length, displacement, or point of traction by which it can make another element larger.

Likewise, no element can make zero larger or increase in value since zero has no length, displacement, or point of traction by which another element can exert force or perform work to make it larger or increase in value.

Since multiplication is unable to do anything to make zero larger or increase in value while addition is, multiplication needs a different identity element.

However, while multiplication is unable to change zero, multiplication by zero performs work in the sense its product rejects the value or numeric information that is contained in the other element since zero reflects the absence of an element.

In other words, for multiplication to do work in increasing the size or value of its elements in a shared product, its elements must share a common boundary, determined by sharing a common trait or characteristic such as a measurement of length or distance, which may be used to calculate area.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

But since zero reflects the absence of an element, it does not share a boundary in common with any other element. As a result, multiplication by zero is unable to do any work other than to return a value of zero as an indication that the multiplication was unable to do any work to make its product larger or increase in size.

From another point of view, since a basic operation such as addition uses a base element to create elements within a single set and processes the information about those elements within the same set or over a single dimension, its identity element operates within a single set or dimension.

But as seen in the additive array, a higher order operation such as multiplication operates over two sets. As a result, its identity element must operate over two sets or dimensions. A dimension may generally be viewed as a set of elements, which are ordered in space or time along a line or curve, are consistent with each other, and usually independent of other dimensions.

From another point of view, as the identity element for multiplication, one cannot contain any part of zero since multiplication by zero effectively destroys the numeric information of another element. Otherwise, one would be unable to retain the numeric information about another element exactly.

Likewise, as the identity element for addition, zero cannot contain any part of one since one is the base element addition uses to generate the set of natural numbers. Otherwise, zero would perform work in calculating another element, and would no longer be its identity element.

Another property of zero is that a multiplication product of zero requires the use of zero as one of its elements, just as only a prime number can make a product of prime numbers.

In summary, where multiplication by zero destroys the value or numeric information of an element, multiplication by one retains it exactly. Where zero may be characterized as neutral regarding the value or numeric information of another element, one is able to serve as the base element that creates numeric values, using addition.

DIVISION BY ZERO

Where multiplication by zero is accepted, division by zero is prohibited since it is unable to reverse the work that is done by multiplication by zero. In other words, since division by zero is unable to recapture the numeric value or identity of the element multiplied by zero, its result is unspecified, and so the operation is prohibited.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

From another point of view, division by zero is like asking what number when multiplied by zero results in a given number, other than zero. Since the product of multiplication by zero is always zero, there is no number that when multiplied by zero results in a number other than zero, so there is no reverse operation using zero. Or, since division by zero does not calculate a specific element, the operation is prohibited.

In other words, where an operation usually results in an element that is clearly identified, division by zero does not provide an algorithm that is able to identify a specific element. Since its result is unspecified, division by zero loses the character of being an operation between two elements, and so is prohibited.

However, multiplication by zero is allowed since it results in a specific element that is clearly identified, namely zero. Likewise, zero divided by an element other than zero is allowed since it results in a specific element that is clearly identified, namely zero.

From a Cartesian point of view, the prohibition against division by zero means that a point of origin cannot be inverted by a reverse operation. A point of origin remains stable even as an operation changes the value or coordinates of other points.

In other words, since a point of origin is fixed or absolute with respect to every point, it remains fixed or absolute in comparisons between points. It stabilizes their coordinates, and appears in comparisons only indirectly, used to determine their coordinates.

To illustrate this idea, a comparison between points a and b may be written using the expression $(a – 0)/(b – 0)$, which displays their coordinates using zero to indicate their point of origin. But using a point of origin as a basis of comparison changes the expression to $(a – 0)/(0 – 0)$.

While algebraically this expression simplifies to $a/0$, $(0 – 0)$ does not equate to a point of origin or 0, but indicates a point of singularity, which does not exist within the coordinate system. In other words, as a point of origin, 0 does not equate to $(0 – 0)$, which algebraically removes a point of origin from its coordinate system.

From another point of view, $(a – 0)/(0 – 0)$, which displays the points as a function of their point of origin, shows that the comparison of point a to its point of origin has a point in common with the denominator and forms the denominator so that the comparison no longer involves two distinct points, a requirement for making comparisons.

As a point of origin, zero is not a void or nothing, but the center of a coordinate system, from which all other points are ordered in terms of distance and direction. It is often used as a point of origin in the measurement of distance or time, or to indicate an initial state or point of balance or equilibrium in physical systems.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

In set theory, zero represents the quantization of nothing, the numeric value of an empty set as an element of a set, which holds the template, or the traits or characteristics that are used to define an element of a set. While zero has no value or quantity since it represents the absence of an object or element of the set, the template allows zero to be included as an element, and interact with other elements.

Since zero represents a set that is empty, lacking any presence or value, it may be considered unique compared to every other element of a set, which enables it to serve as a point of origin, balance, or equilibrium.

In contrast, one represents the quantization of a single object as an element of a set, or unit value. While one does not represent multiple objects or fractions, values for multiple objects, fractions, or parts of one may be generated by using one as a base element in various operations.

Since one represents the quantization of a single object or element as an element of a set, it reflects the presence of an object or element, and represents the opposite of zero, which reflects the absence of an object or element.

MULTIPLICATIVE UNION

Set theory offers another way to show that addition and multiplication require different identity elements. For example, multiplication draws elements from two different sets to define an additive array or calculate an area. While it may draw the elements from just a single set, it uses different sets to calculate its product, which changes the quantity of its elements and their identity as members of a set.

In particular, multiplication forms a bond or multiplicative union (the idea of a multiplicative union is based on a union between two sets) between the two sets it draws its elements from, which imparts information about its product being a member of a new set or subset that represents a combination of the two sets, which, in a geometrical sense, makes it two dimensional.

For example, to calculate area, multiplication draws an element from a set that represents length on a horizontal axis, and an element from a set that represents height on a vertical axis. As a result, its product represents an element of a set that represents area, determined by two displacements in two dimensions.

Alternatively, if the multiplication draws one or both of its elements from a set of abstract numeric multipliers, its identity as an element of a set remains one dimensional, retaining

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

either the identity of the non-abstract numeric element, or the identity of an abstract numeric element.

In contrast, addition does not change the identity of its sum as a member of a set. Instead, addition combines two elements from a set into another element of the set, changing only its quantity. In particular, addition counts the value of an element from a set, and counts the value of another element from the same set to arrive at the sum as an element of the same set, which, in a geometrical sense, makes it one dimensional.

So, where in multiplication the identity element preserves the value of the other element and the identity of both sets, in addition, the identity element preserves only the value of the other element, and does not need to touch upon the identity of the set since addition counts elements over the same set.

As a result, zero, the identity element for addition, is configured to preserve the value of another element while counting over the same set, and does not touch upon the identity of the set. But since multiplication does more than count elements within the same set, it needs an identity element with a different configuration.

Since multiplication registers the value or quantity of its elements and their identity as members of a set, it needs an identity element that is able to register the value or quantity of an element and its identity as a member of a set. Zero is not able to do this since it is neutral about the value or numeric information of another element and is configured to operate over a single set. As a result, zero is unable to count or register the value or quantity of another element, or its identity as a member of a set.

In other words, since zero is assumed to operate over a single set, it is not configured to preserve the value or numeric information of an element over two sets or dimensions, a requirement for the multiplication identity element.

Rather, when multiplication uses zero, it calculates an additive array with a sum of zero or zero sum, or an area or other two dimensional quantity as a point, which changes the configuration of its product from two to one dimensions.

From a geometrical point of view, zero is able to serve as a point of origin since it represents a point that lacks any displacement among a set of points or coordinate system. While in addition zero preserves the displacement of another point, in multiplication zero is unable to preserve a displacement or calculate an area with a value other than itself, although its product is able to retain the units of area.

In other words, as a point of origin, zero remains a point of origin that lacks any displacement in multiple dimensions.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

In contrast, multiplication by one allows displacements to be created with a uniform displacement in multiple dimensions, which counts and preserves the value of another element and its identity as a member of a set exactly.

SYMMETRY

Identity elements may be characterized by the type of symmetry they are associated with as a point of origin used for organizing or ordering the elements of a set, or as a point of reference or intersection used for comparing elements. Symmetry introduces different requirements for an identity element.

For example, to order the elements of a set, an identity element must be independent of the other elements of the set so its presence does not affect their ordering. In other words, it must possess a trait or characteristic, or point or region of no intersection it does not share in common with any element.

One way for an identity element to possess a point or region of no intersection with all the other elements of a set is to lie outside the set that it orders. By lying outside a set, an identity element will possess a point or region of no intersection that it does not share in common with any element, letting it serve as a point of origin that is fixed or absolute with respect to all the other elements of the set.

From another point of view, since ordering a set encloses all the elements of the set and gives each element a distinctive piece of information about its order, typically associated with its size, the identity element for ordering the set must lie outside the set so it does not affect the ordering of its elements.

But while it lies outside the set, the identity element for ordering the set must possess traits or characteristics, or a template that is compatible with the elements of the set so it can interact with them.

To accommodate this contradiction where an identity element lies outside the set it orders, but possesses traits or characteristics that are compatible with the elements of the set, the identity element may reflect the absence of an element so it lies outside the set and does not affect the ordering of its elements while possessing traits or characteristics that are compatible with the elements of the set.

This requirement for an identity element that represents the absence of an element is virtually the same requirement for a point of origin, which does not possess any length or displacement in a system of coordinates. While any point possesses this property, all the

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

other points in a coordinate system possess a length or displacement compared to a point of origin.

In fact, since any point is quantized with a minimum length or displacement, a point of origin is distinguished as the central point of a coordinate system, from which length or displacement is computed as a discrete amount, consistent with the quantized length or displacement of another point. As the center of a coordinate system, a point of origin is quantized with no length or displacement to determine the coordinates of other points.

In contrast, the second type of symmetry requires an identity element that shares a trait or characteristic, or point or region of intersection in common with every element of the set. Since comparisons between different elements are based on the transfer of information about a common trait or characteristic, the second type of symmetry requires an identity element that possesses a trait or characteristic, or point or region of intersection in common with every element.

If a set is able to be generated from a base element or its seed, just as the set of rational numbers and irrational numbers are able to be generated using the set of natural numbers as a seed set, this requirement for an identity element for the second type of symmetry may be met by using the base element for the seed set.

In other words, when a set like the set of natural numbers may be generated from a base element, that element provides a point of intersection in common with all the elements of the set, and other sets generated from it. It gives all the elements of the sets a common point of intersection for making comparisons.

In contrast, the identity element for the first type of symmetry, which possesses a point or region of no intersection that it does not share in common with any element of the set, is unable to be used in making comparisons since comparisons require a common point of intersection to transfer information about different elements.

While the identity element for the first type of symmetry could possess a point or region in common with every element as well as a point or region of no intersection that it does not share with any element, since it lies outside the set that it orders it does not have a point or region in common with every element. In other words, since it reflects the absence of an element, it lacks a point or region in common with every element.

Moreover, since the identity element for the second type of symmetry shares a point or region in common with every element, it cannot be an element of the set if it also shares a point or region of no intersection with all the elements of the set. An element must be distinct, not a subset that is comprised of an element that lies within the set and an element that lies outside the set.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

In other words, since the second type of symmetry requires a common point of intersection, it precludes using the identity element for the first type of symmetry since a comparison requires the presence, and not the absence, of an object or element for its identity element.

From another point of view, elements may be compared only after their order is established since ordering identifies each element, letting it be associated with a common trait or characteristic such as size. As a result, the identity element for ordering a set is independent of the identity element used for comparing elements, and likely to be a different element.

Moreover, since comparisons between elements are based on information about a common trait or characteristic, which is typically associated with their order or size, the identity element for an operation that involves a common trait or characteristic, such as size, is independent of the identity element used for their ordering.

This distinction between ordering the elements of a set and making comparisons between them is virtually the same distinction between using addition to generate the set of natural numbers and using division to compare two numbers in terms of their relative size or numeric value.

To construct the identity element for the first type of symmetry for a set generated by a base element and basic operation, a new operation needs to be introduced that imparts a different sense of direction or orientation than possessed by all the elements of the set.

This requirement for a new operation to calculate the identity element may be met by using the reverse operation of the basic operation since it imparts a different sense of direction or reverse orientation to an element. For addition over the set of natural numbers, this requirement may be met by using subtraction.

In other words, subtraction is able to calculate the identity element for the first type of symmetry. This may be done by subtracting an element from itself to determine the identity element as a point of balance or equilibrium.

In other words, where a basic operation is able to use a base element and a duplicate of it to generate new elements, a reverse operation is able to both reverse the work of the basic operation and calculate its identity element as a point of balance or equilibrium, by using a similar format that applies the reverse operation to an element that is duplicated, or using an equation such as a – a = identity element.

As an aside, since inserting any element into the equation a – a = identity element gives the same result, this suggests that this identity element may have the potential to destroy the

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13

identity of other elements in higher order operations, which process the information about an element in a different manner.

If the set of natural numbers and negative integers are ordered on a line, the identity element for the first type of symmetry appears as a point of balance or equilibrium lying between the two sets, which is not an element of either set since it possesses a different sense of direction or orientation than all the elements of either set.

This new element is zero. Zero satisfies the new requirement for an identity element for the set of natural numbers as an element that lies outside the set, and possesses a different sense of direction or orientation than all the elements of the set. Yet it shares many of the same traits or characteristics since it may be constructed from any natural number, which gives it the same elemental composition or spacing on a number line, and is ordered consecutively, as it precedes one.

Zero may also be constructed as a common point of origin between the set of natural numbers and negative integers by adding an element from the set of natural numbers with its mirror image from the set of negative integers.

The only difference between zero and the natural numbers is that zero possesses a different sense of direction or orientation, and its numeric value reflects the absence of an object or element, or the quantization of nothing, just as a point of origin does not possess a length or displacement apart from itself.

Since zero represents the absence of an object or element, or the quantization of nothing, it is unable to be used as a point of reference or an identity element for the second type of symmetry that involves comparisons, as comparisons require the presence, not the absence of an element and information about its size.

Moreover, zero is unable to be used to generate other elements since the absence of an object or element is unable to produce the presence of an object or element, and this trait of zero holds whether it is used in addition, subtraction, or higher order operations such as multiplication and square roots.

In contrast, since one represents the quantization of a single object or element, it can be used to generate other elements in addition, subtraction, and other operations, and is able to serve as the identity element for the second type of symmetry since it retains information about the order or size of an element exactly.

In a sense, as a point of origin, zero can be used to answer the question of where I am in space and time, while one is used as a point of reference to answer the question of who I am in terms of comparing myself to others.

From another point of view, if a single element is able to generate a set, its identity element requires the construction of a new element outside the set. But if a set is not able to be generated using a base element and operation, one of its elements may serve as its identity element just as one is able to serve as the identity element for multiplication since multiplication is unable to generate the set of natural numbers using one.

In other words, the base element that is used to generate a set of elements is unable to serve as the identity element for the operation used to generate the set, but may serve as the identity element for higher order operations. Likewise, the identity element for a basic operation that generates a set is unable to serve as the identity element for higher order operations.

In other words, basic and higher order operations require different identity elements due to the difference between ordering the elements in a set and making comparisons between them. One identity element serves as a point of origin for ordering elements. The other identity element, the base element used to generate elements, serves as a common point of intersection or reference for comparing elements.

Because of this independence between the base element used to construct other elements and the identity element for a basic operation, the identity element for the basic operation is constructed using the reverse operation of the basic operation.

CONCLUSION

As a result, the following theorem may be considered proven:

For a set generated using a base element and basic operation, the identity element for that set and operation requires the construction of a new element that has a different sense of direction or orientation than all the elements of the set, which may be satisfied by using the reverse operation of the basic operation.

For higher order operations, the base element that is used to construct the elements of a set may serve as the identity element since it defines a common point of intersection or reference for comparing elements.

ACKNOWLEDGMENT

I gratefully acknowledge the kind invitation to publish by the editor of the International Journal of Arts and Science Research, Arun Haris, and his patience in letting me prepare the article to the journal’s standards.

CONFLICT OF INTEREST

I declare I have no conflict of interest.

BIBLIOGRAPHY

1. "Logical Equivalence Failure," James H. Hughes, *International Journal of Engineering and Robot Technology*, Volume 3, Number 2, 2016, pp. 58-72.

The key paragraphs in this article that prompted the proposition of how addition and multiplication require different identity elements are found in the section titled "Elements of Mathematics," as follows.

As a rule, logic seeks consistency. This drive for consistency sometimes results in rules that may seem arbitrary such as the rule of algebra that prohibits division by zero or any algebraic expression such as (x – x) that is equal to zero. Otherwise, division by zero lets any number or algebraic expression become equal to each other when they are clearly different.

While the rule that prohibits division by zero is not arbitrary since the operation of division requires a number with value, zero has a prominent role in the number system as the identity element of addition and subtraction, and balance point between the positive and negative numbers [integers].

2. *Topics in Algebra*, I. N. Herstein, Ginn and Company, 1964.

This textbook was used in the two classes on abstract algebra taken by the author while attending the University of Texas at Austin in the 1970's. It provides some background on set theory, and delineates group and ring theory.

3. Author's education. The author attended the University of Texas at Austin (B.S. in mathematics in 1977, and MBA in 1978 under the actuarial science program). This gave him a strong background in applied mathematics. But this education did not include the application of set theory to explain addition and multiplication, or why they require different identity elements.

@ ECRTD-UK https://www.eajournals.org/
Journal level DOI: https://doi.org/10.37745/ijmss.13